Compressible Fluid Dynamics Past Papers

By Shikang Ni

Contents

1 2018-19 (MOCK)

1.1 Q1

1. (Part I)

(a)

$$u_t + uu_x = 0 \quad , \qquad \text{with} \quad u(x,0) = f(x_0)$$
$$\text{LHS} = f'(x-ut)\cdot(-u) + u\cdot f'(x-ut) = 0 = \text{RHS}$$

(b) Characteristic lines are lines along which characteristic variables are constant, $\frac{d\nu}{ds} = 0$.

$$dx = f(x_0)dt$$

The solution along these curves have eigenvalue of $f(x_0)$, which is a constant.

(c)

$$\text{Solve: } \begin{cases} x - x_0 = tf(x_0) \\ x - (x_0+\epsilon) = tf(x_0+\epsilon) \end{cases}$$

$$tf(x_0) - \epsilon = tf(x_0+\epsilon)$$

$$\lim_{\epsilon\to 0} \frac{f(x_0+\epsilon) - f(x_0)}{\epsilon} = -\frac{1}{t} = f'(x_0)$$

So,

$$t = -\frac{1}{f'(x_0)} \quad \text{and} \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

(d) If $f'(x_0) \geq 0$, nearby lines separated by ϵ will meet at $t < 0$. The characteristic lines never meet so the solution is single valued. The characteristic is expansive.

(e) If $f'(x) < 0$, the characteristics meet at $\min(t) = \frac{1}{\max(|f'(x_0)|)}$

(Part II)

(a) The CFL coefficient describes the ratio between the maximum physical wave speed to maximum speed of information travel on the grid:

$$CFL = |a|\frac{\Delta t}{\Delta x}$$

It is used for calculating the maximum stable timestep for the simulation.

(b) Taylor expansion: $f(x+\Delta x) = f(x) + \frac{\Delta x}{1}\frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 f}{\partial x^2} + ...$

$$u_i^{n+1} = u_i^n + \frac{\partial u_i^n}{\partial t}\Delta t + \frac{(\Delta t)^2}{2}\frac{\partial^2 u_i^n}{\partial t^2} + \mathcal{O}(\Delta t^3)$$

$$u_{i-1}^n = u_i^n - \frac{\partial u_i^n}{\partial x}\Delta x + \frac{(\Delta x)^2}{2}\frac{\partial^2 u_i^n}{\partial x^2} - \mathcal{O}(\Delta x^3)$$

$$u_{i+1}^n = u_i^n + \frac{\partial u_i^n}{\partial x}\Delta x + \frac{(\Delta x)^2}{2}\frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}(\Delta x^3)$$

(c)

$$\text{Subtracting: } \partial_x u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \mathcal{O}(\Delta x^2)$$
$$\text{Adding: } \partial_{xx} u_i^n = \frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Note:

$$(\partial_t + a\partial_x)u = 0$$
$$(\partial_t - a\partial_x)(\partial_t + a\partial_x)u = 0$$
$$\Rightarrow (\partial_{tt} - a^2\partial_{xx})u = 0$$

Continuing:

$$(\partial_{tttt} - a^4\partial_{xxxx})u = 0$$

Cannot cancel third order.

$$\begin{aligned}
u_i^{n+1} &= u_i^n + \frac{\partial u_i^n}{\partial x}\Delta t + \frac{1}{2}\Delta t^2 \frac{\partial^2 u_i^n}{\partial t^2} \\
&= u_i^n - (a\Delta t)\partial_x u_i^n + \left(\frac{1}{2}a^2\Delta t^2\right)\partial_{xx} u_i^n + \mathcal{O}(\Delta t^3) \\
&= u_i^n - \frac{1}{2}\frac{a\Delta t}{\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}\frac{a^2\Delta t^2}{\Delta x^2}(u_{i+1}^n + u_{i-1}^n - 2u_i^n) + \mathcal{O}(\Delta x^2, \Delta t^3) \\
&= u_i^n - \frac{1}{2}c(u_{i+1}^2 - u_{i-1}^n) + \frac{1}{2}c^2(u_{i+1}^n + u_{i-1}^n - 2u_i^n) + \mathcal{O}(\Delta x^2, \Delta t^3) \\
&= \left(\frac{1}{2}c^2 + \frac{1}{2}c\right)u_{i-1}^n + \left(\frac{1}{2}c^2 - \frac{1}{2}c\right)u_{i+1}^n + (1-c^2)u_i^n + \mathcal{O}(\Delta x^2, \Delta t^3) \\
&= \frac{1}{2}c(c+1)u_{i-1}^n - \frac{1}{2}c(1-c)u_{i+1}^n + (1-c^2)u_i^n
\end{aligned}$$

where $c = \frac{a\Delta t}{\Delta x}$

(d) Second order in space and third order in time?

(e) i. Maintains the bell-shape, but dispersive effects (phase error) tend to skew the shape.

ii. Oscillations produced at the edge of discontinuities.

(f)

$$\int_0^{\Delta t}\int_{x-1/2}^{x+1/2} u_t \, dx dt + \int_0^{\Delta t}\int_{x-1/2}^{x+1/2} f(u)_x \, dx dt = 0$$

$$\int_{x-1/2}^{x+1/2} u(x,\Delta t) - u(x,0)\, dx + \int_0^{\Delta t} f(u(x+1/2,t)) - f(u(x-1/2),t)\, dt = 0$$

Define cell average:

$$u_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t^n)\, dx \quad \text{and} \quad f_{i+1/2} = \frac{1}{\Delta t}\int_0^{\Delta t} f(x_{i+1/2},t)\, dt$$

$$\Delta x u_i^{n+1} = \Delta x u_i^n - \Delta t[f_{i+1/2} - f_{i-1/2}]$$

For linear advection with $a > 0$, $f_{i+1/2} = au_i$. Hence,

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}(u_i - u_{i-1})$$

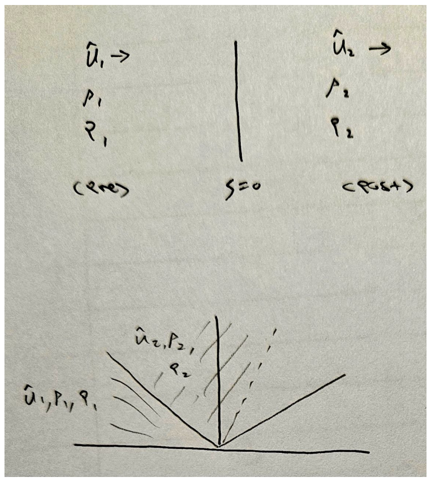
1.2 Q2

2. (Part I)

(a) A Riemann problem is an initial value problem consisting of a conservation equation and two piece-wise constant initial states separated by a single discontinuity at $x = x_0$. It has non-trivial exact solution.

(b)

$$\rho_1 \hat{u}_1 = \rho_2 \hat{u}_2$$
$$\rho_1 \hat{u}_1^2 + p_1 = \rho_2 \hat{u}_2^2 + p_2$$
$$\hat{u}_1(\rho_1 E + p_1) = \hat{u}_2(\rho_2 E + p_2)$$



(c) Across a contact wave, pressure and velocity is constant, but the density jumps. We can just use $\{\rho_L^*, \rho_R^*, u^*, p^*\}$

(d)

$$\rho_L(u_L - S) = \rho_L^*(u^* - S) \Rightarrow \hat{Q}_L = \rho_L^*(u^* - S)$$
$$\rho_L(u_L - S)^2 + p_L = \rho_L^*(u^* - S)^2 + p^*$$

$$\hat{Q}_L(u_L - S) + p_L = \hat{Q}_L(u^* - S) + p^*$$
$$\hat{Q}_L(u_L - u^*) = p^* - p_L$$
$$u^* = u_L - \frac{p^* - p_L}{\hat{Q}_L}$$

(Part II)

(a) The FORCE flux is the average of the Lax-Friedrichs flux and Richtmyer flux.

$$\mathbf{F}_{i+1/2}^{FORCE} = \frac{1}{2}(\mathbf{F}_{i+1/2}^{LF} + \mathbf{F}_{i+1/2}^{RI})$$

FORCE scheme is:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+1/2}^{FORCE} - \mathbf{F}_{i-1/2}^{FORCE}\right)$$

LF is first-order, RI is second-order, FORCE is first-order.

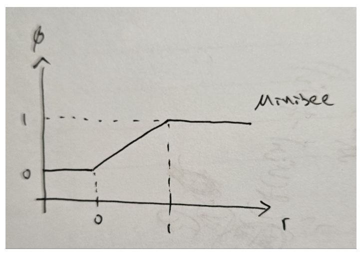
(b) The FORCE scheme is a first-order method, which capture discontinuities well but suffer from greater diffusion. The Richtmyer method is a second-order method that is more accurate in smooth areas but suffer from oscillations near discontinuities. Combining those two methods can lead to a high resolution method which is second-order everywhere except near discontinuities, where it reverts to the first-order method. The process of combining fluxes from different method is called flux limiting. The resulting scheme is a high resolution TVD method.

$$f_{i+1/2}^{FLIC} = f_{i+1/2}^{FORCE} + \phi_{i+1/2}\left(f_{i+1/2}^{RI} - f_{i+1/2}^{FORCE}\right)$$

where F^{FORCE} is first-order monotone scheme. f^{RI} is a high-order scheme. $\phi_{i+1/2}$ is the limiter function. The method is called flux limiter centred scheme (FLIC). Then,

$$u_i^{n+1} = u_t^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^{FLIC} - f_{i-1/2}^{FLIC}\right)$$

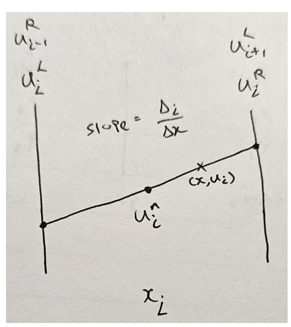
An example of the limiter function is the minibee:



r is the a measure of the change of slope:

$$r = \frac{\Delta u_{upw}}{\Delta u_{downw}}$$

(c) SLIC is the slope limited centred scheme. Slope limiting methods use first-order monotone schemes but moves beyond piecewise-constant approximations to piecewise-linear approximations for the data in a cell . It consists of 3 steps:



Step 1: Linear reconstruction: fitting a linear function through $\boldsymbol{u}_i^n$ using information of neighbouring slopes:

$$\mathbf{u}_i(x) = \mathbf{u}_i^n + (x - x_i)\frac{\boldsymbol{\Delta}_i}{\Delta x}$$

where $\boldsymbol{\Delta}_i$ is a measure of the slope, given by combination of neighbouring slopes:

$$\boldsymbol{\Delta}_i = \frac{1}{2}(1+\omega)\boldsymbol{\Delta}_{i-1/2} + \frac{1}{2}(1-\omega)\boldsymbol{\Delta}_{i+1/2} \quad , \quad \omega \in [-1, 1]$$

where: $\boldsymbol{\Delta}_{i-1/2} = \boldsymbol{u}_i^n - \boldsymbol{u}_{i-1}^n$ and $\boldsymbol{\Delta}_{i+1/2} = \boldsymbol{u}_{i+1}^n - \boldsymbol{u}_i^n$.

Hence, at the cell boundary, we have $x = x_i - \frac{1}{2}\Delta x$ (L) and $x = x_i + \frac{1}{2}\Delta x$ (R):

$$\boldsymbol{u}_i^L = \boldsymbol{u}_i^n - \frac{1}{2}\boldsymbol{\Delta}_i \quad \text{and} \quad \boldsymbol{u}_i^R = \boldsymbol{u}_i^n + \frac{1}{2}\boldsymbol{\Delta}_i$$

We can perform slope limiting using the slope limiter function $\xi(r)$:

$$\bar{\boldsymbol{u}}_i^L = \boldsymbol{u}_i^n - \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i \quad \text{and} \quad \bar{\boldsymbol{u}}_i^R = \boldsymbol{u}_i^n + \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i$$

Step 2: Half-time step evolution of boundary states in $V = [x_L, x_R] \times [n, n+1/2]$:

$$\bar{\boldsymbol{u}}_i^{L,n+1/2} = \bar{\boldsymbol{u}}_i^L - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\boldsymbol{f}(\bar{\boldsymbol{u}}_i^R) - \boldsymbol{f}(\bar{\boldsymbol{u}}_i^L)\right)$$

$$\bar{\boldsymbol{u}}_i^{R,n+1/2} = \bar{\boldsymbol{u}}_i^R - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\boldsymbol{f}(\bar{\boldsymbol{u}}_i^R) - \boldsymbol{f}(\bar{\boldsymbol{u}}_i^L)\right)$$

Step 3: Compute flux with FORCE scheme and perform finite volume update:

$$\boldsymbol{f}_{i+1/2}^{SLIC} = \boldsymbol{f}_{i+1/2}^{FORCE}\left(\bar{\boldsymbol{u}}_i^{R,n+1/2}, \bar{\boldsymbol{u}}_{i+1}^{L,n+1/2}\right) = \frac{1}{2}\left(\boldsymbol{f}_{i+1/2}^{LF} + \boldsymbol{f}_{i+1/2}^{RI}\right)$$

where

$$\boldsymbol{f}_{i+1/2}^{LF} = \frac{1}{2}\left(\boldsymbol{f}(\bar{\boldsymbol{u}}_i^{R,n+1/2}) + \boldsymbol{f}(\bar{\boldsymbol{u}}_{i+1}^{L,n+1/2})\right) + \frac{1}{2}\frac{\Delta x}{\Delta t}\left(\bar{\boldsymbol{u}}_i^{R,n+1/2} - \bar{\boldsymbol{u}}_{i+1}^{L,n+1/2}\right)$$

and

$$\boldsymbol{f}_{i+1/2}^{RI} = \boldsymbol{f}(\boldsymbol{u}_{i+1/2}^{n+1/2})$$

$$\boldsymbol{u}_{i+1/2}^{n+1/2} = \frac{1}{2}\left(\bar{\boldsymbol{u}}_i^{R,n+1/2} + \bar{\boldsymbol{u}}_{i+1}^{L,n+1/2}\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\boldsymbol{f}(\bar{\boldsymbol{u}}_i^{R,n+1/2}) - \boldsymbol{f}(\bar{\boldsymbol{u}}_{i+1}^{L,n+1/2})\right)$$

Update according to:

$$\boldsymbol{u}_i^{m+1} = \boldsymbol{u}_i^n - \frac{\Delta t}{\Delta x}\left(\boldsymbol{f}_{i+1/2}^{SLIC} - \boldsymbol{f}_{i-1/2}^{SLIC}\right)$$

(d) Flux limiting combine different flux methods of different orders and activate them at appropriate regions of the solution using limiter function. Slope limiting aims to better approximate boundary states with piecewise-linear functions instead of just piecewise-constants. SLIC is less diffusive.

1.3 Q3

3. (a)

$$\begin{vmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \rho a^2 & u-\lambda \end{vmatrix} = 0$$

$$(u-\lambda)[(u-\lambda)^2 - a^2] = 0 \quad \Rightarrow \quad \lambda = u, u+a, u-a$$

$$\Lambda = \begin{pmatrix} u-a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u+a \end{pmatrix}$$

The eigenvalues are real and unique, so it is hyperbolic.

(b) $Ar = \lambda r$

For $\lambda_0 = u$,

$$\begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{cases} uv_1 + \rho v_2 = uv_1 \\ uv_2 + v_3/\rho = uv_2 \\ \rho a^2 v_2 + uv_3 = uv_3 \end{cases}$$

$$v_1 = k, v_2 = v_3 = 0$$

$$\boldsymbol{r}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_- = u - a$,

$$\begin{cases} \rho v_2 = -av_1 \Rightarrow v_2 = -(a/\rho)v_1 \\ v_3/\rho = -av_2 \\ \rho a^2 v_2 = -av_3 \Rightarrow v_3 = -\rho a v_2 \end{cases}$$

$$\boldsymbol{r}_- = \begin{pmatrix} 1 \\ -a/\rho \\ a^2 \end{pmatrix}$$

For $\lambda_+ = u + a$,

$$\begin{cases} \rho v_2 = -av_1 \Rightarrow v_2 = (a/\rho)v_1 \\ v_3/\rho = av_2 \\ \rho a^2 v_2 = -av_3 \Rightarrow v_3 = \rho a v_2 \end{cases}$$

$$\boldsymbol{r}_+ = \begin{pmatrix} 1 \\ a/\rho \\ a^2 \end{pmatrix}$$

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 1 & 1 \\ -a/\rho & 0 & a/\rho \\ a^2 & 0 & a^2 \end{pmatrix}$$

$lA = \lambda l$

For $\lambda_0 = u$

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{pmatrix} = u \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$\begin{cases} uv_1 = uv_1 \\ \rho v_1 + uv_2 + \rho a^2 v_3 = uv_2 \\ v_2/\rho + uv_3 = uv_3 \end{cases}$$

$$v_1 = -a^2 v_3, v_2 = 0$$

$$\boldsymbol{l}_0 = \begin{pmatrix} 1 & 0 & -1/a^2 \end{pmatrix}$$

For $\lambda_- = u - a$

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{pmatrix} = (u-a) \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$\begin{cases} uv_1 = (u-a)v_1 \Rightarrow v_1 = 0 \\ \rho v_1 + uv_2 + \rho a^2 v_3 = (u-a)v_2 \\ v_2/\rho + uv_3 = (u-a)v_3 \Rightarrow v_3 = -\frac{1}{\rho a} v_2 \end{cases}$$

$$\boldsymbol{l}_- = \begin{pmatrix} 0 & 1 & -\frac{1}{\rho a} \end{pmatrix}$$

For $\lambda_+ = u + a$

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{pmatrix} = (u+a) \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$\begin{cases} v_1 = 0 \\ \rho v_1 + uv_2 + \rho a^2 v_3 = (u+a)v_2 \\ v_2/\rho + uv_3 = (u+a)v_3 \Rightarrow v_3 = \frac{1}{\rho a} v_2 \end{cases}$$

$$\boldsymbol{l}_+ = \begin{pmatrix} 0 & 1 & \frac{1}{\rho a} \end{pmatrix}$$

$$\boldsymbol{Q}^{-1} = \begin{pmatrix} 0 & 1 & -1/\rho a \\ 1 & 0 & -1/a^2 \\ 0 & 1 & 1/\rho a \end{pmatrix}$$

(c)

$$\begin{pmatrix} d\nu_- \\ d\nu_0 \\ d\nu_+ \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1/\rho a \\ 1 & 0 & -1/a^2 \\ 0 & 1 & 1/\rho a \end{pmatrix} \begin{pmatrix} d\rho \\ du \\ dp \end{pmatrix} = \begin{pmatrix} du - dp/\rho a \\ d\rho - dp/a^2 \\ du + dp/\rho a \end{pmatrix}$$

(d) Characteristic wavefronts: $\frac{d\boldsymbol{\nu}}{ds} = 0$

$$\frac{\partial \boldsymbol{\nu}}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial \boldsymbol{\nu}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial s} = 0$$

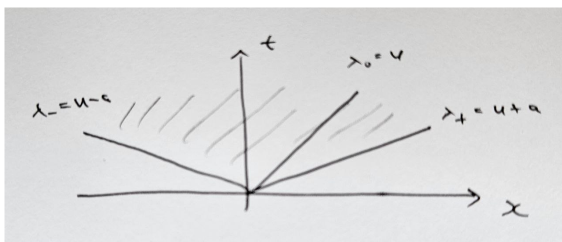
$$\Rightarrow t = s \quad \text{and} \quad \frac{\partial \boldsymbol{x}}{\partial t} = \Lambda$$

Along the characteristic lines, the corresponding characteristic variable is constant

$$v_+ = \int du + \frac{dp}{\rho a} = \text{const} \quad \text{along} \quad dx = (u+a)dt$$

$$v_- = \int du - \frac{dp}{\rho a} = \text{const} \quad \text{along} \quad dx = (u-a)dt$$

(e) The domain of dependence (backwards) and influence (forward) of the Euler equation is determined by the slowest and fastest characteristics and is always a bounded interval.



(f) **Sod test problem**: we consider a tube with membrane in the middle, with high pressure material on the left of the membrane and low pressure material on the right of the membrane. At $t = 0$, we remove the membrane and let the flow evolve. We expect to see a rightward moving shock wave, a leftward moving rarefaction wave and a contact discontinuity in between. The contact corresponds to the location of the material interface.

(g) We can perform the update along one direction first, then use the result to update the other direction. For example, we can update along x-direction first then y-direction. For example, we could do a first-order update in the x-direction followed by y-direction:

$$\boldsymbol{u}^{n+1} = \mathcal{L}^y \mathcal{L}^x(\boldsymbol{u}^n)$$

$$\bar{\boldsymbol{u}}_{i,j} = \boldsymbol{u}^n_{i,j} + \frac{\Delta t}{\Delta x}\left(\boldsymbol{f}^n_{i-1/2,j}(\boldsymbol{u}) - \boldsymbol{f}^n_{i+1/2,j}(\boldsymbol{u})\right)$$

$$\boldsymbol{u}^{n+1}_{i,j} = \bar{\boldsymbol{u}}_{i,j} + \frac{\Delta t}{\Delta y}\left(\boldsymbol{g}^n_{i,j-1/2}(\bar{\boldsymbol{u}}) - \boldsymbol{g}^n_{i,j+1/2}(\bar{\boldsymbol{u}})\right)$$

$$\Delta t = C \times \min\left(\frac{\Delta x}{a_{max,x}}, \frac{\Delta y}{a_{max,y}}\right)$$

The stable time-step is halved.

(h) Operator splitting.

2 2019-20 (MOCK)

2.1 Q1

1. (Part I)

(a) Characteristics - lines along which the characteristic variables are constant.

$$\frac{\mathrm{d}\nu}{\mathrm{d}s} = \frac{\mathrm{d}\nu}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\mathrm{d}\nu}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = 0 \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}t} + a\frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

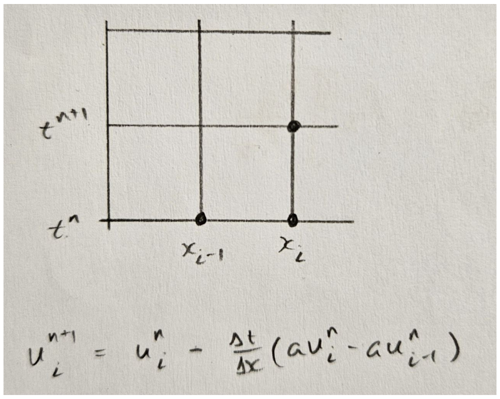
$$\Rightarrow s = t \quad \text{and} \quad dx = adt$$

The characteristics are:

$$x = x_0 + at$$

They are straight line in the x-t plane.

(b)



$$u_t = \frac{u_i^{n+1} - u_i^n}{\Delta t} \quad \text{and} \quad u_x = \frac{u_i^n - u_{i-1}^n}{\Delta x}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) \qquad (a > 0)$$

(c) The CFL coefficient, $C = \frac{a_{\max}}{\Delta x/\Delta t}$, is the ratio of maximum physical wave speed to the maximum speed of information travel on the grid. It is used to calculate the maximum stable time step for the simulation. For this case:

$$0 \leq C \leq 1$$

(d)

$$u_i^{n+1} = u_i^n + \frac{\partial u_i^n}{\partial t}\Delta t + \frac{\partial^2 u_i^n}{\partial t^2}\frac{\Delta t^2}{2} + \mathcal{O}(\Delta t^3)$$

$$u_{i-1}^n = u_i^n - \frac{\partial u_i^n}{\partial x}\Delta x + \frac{\partial^2 u_i^n}{\partial x^2}\frac{\Delta x^2}{2} + \mathcal{O}(\Delta x^3)$$

Modified equation:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

is actually equivalent to:

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2} + a\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial x^2}\frac{\Delta x}{2} + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) = 0$$

Noting that:

$$(\partial_t - a\partial_x)(\partial_t + a\partial_x)u = 0$$
$$(\partial_{tt} - a^2\partial_{xx})u = 0$$

Then,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \left(\frac{a\Delta x}{2} - \frac{a^2\Delta t}{2}\right)\frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{1}{2}a\Delta x(1-c)\frac{\partial^2 u}{\partial x^2}$$

It is of the form: $\partial_t u + a\partial_x u = \alpha\partial_{xx}u$. The unseen term introduces diffusion of the numerical method.

(e)

$$u_i^{n+1} - u_i^n = \frac{\partial u}{\partial t}\Delta t$$
$$\Rightarrow \frac{\partial u}{\partial t} = \frac{1}{\Delta t}(u_i^{n+1} - u_i^n)$$

and

$$\begin{cases} u_{i+1}^n = u_i^n + \frac{\partial u}{\partial x}\Delta x \\ u_{i-1}^{n+1} = u_{i-1}^n + \frac{\partial u_{i-1}}{\partial t}\Delta t = u_i^n - \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial t}\Delta t \end{cases}$$

Subtracting one from the other,

$$u_{i+1}^n - u_{i-1}^{n+1} = 2\Delta x\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\Delta t$$

Cancelling the temporal,

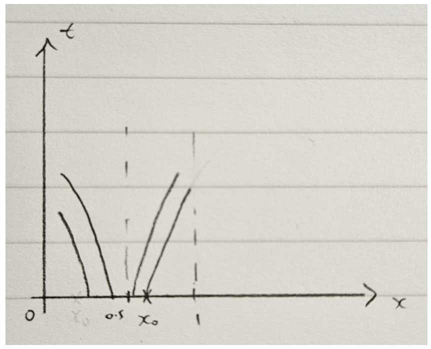
$$(u_{i+1}^n - u_{i-1}^{n+1}) + (u_i^{n+1} - u_i^n) = 2\Delta x\frac{\partial u}{\partial x}$$
$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{2\Delta x}(u_{i+1}^n - u_{i-1}^{n+1}) + \frac{1}{2\Delta x}(u_i^{n+1} - u_i^n)$$

The linear advection update scheme is then:

$$\left(\frac{1}{\Delta t} + \frac{a}{2\Delta x}\right)(u_i^{n+1} - u_i^n) + \frac{a}{2\Delta x}(u_{i+1}^n - u_{i-1}^{n+1}) = 0$$

(Part II)

(a) The characteristics are no longer straight lines, but are now curves.



(b) First-order approximation for the derivative, where forward difference is used when $x \geq \frac{1}{2}$ and backward difference is used when $x < \frac{1}{2}$.

$$\begin{cases} \frac{u_i^{n+1}-u_i^n}{\Delta t} + \left(x_i - \frac{1}{2}\right) \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0 & , x \geq \frac{1}{2} \\ \frac{u_i^{n+1}-u_i^n}{\Delta t} + \left(x_i - \frac{1}{2}\right) \frac{u_{i+1}^n - u_i^n}{\Delta x} = 0 & , x < \frac{1}{2} \end{cases}$$

$$\begin{cases} u_i^{n+1} = u_i^n - \left(x_i - \frac{1}{2}\right) \frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) & , x \geq \frac{1}{2} \\ u_i^{n+1} = u_i^n - \left(x_i - \frac{1}{2}\right) \frac{\Delta t}{\Delta x}(u_{i+1}^n - u_i^n) & , x < \frac{1}{2} \end{cases}$$

(c) CFL condition:

$$0 \leq C \leq 1$$

where

$$C = \frac{|x_i - \frac{1}{2}|}{\Delta x / \Delta t}$$

2.2 Q2

2. (a) Primitive variable form.

(b)

$$\boldsymbol{u} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}$$

(c)

$$\begin{vmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \rho a^2 & u-\lambda \end{vmatrix} = 0$$

$$(u-\lambda)[(u-\lambda)^2 - a^2] = 0$$

Eigenvalues:

$$\lambda = u, u+a, u-a$$

The eigenvalues are real, so the PDE is hyperbolic.

(d) Right eigenvector: $A\boldsymbol{r} = \lambda \boldsymbol{r}$.

$$\begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \lambda \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

For $\lambda_1 = u$,

$$\begin{cases} ux + \rho y = ux \\ uy + (1/\rho)z = uy \\ \rho a^2 y + uz = uz \end{cases}$$

$$\boldsymbol{r_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = u + a$,

$$\begin{cases} \cancel{ux} + \rho y = (\cancel{u} + a)x \\ \cancel{uy} + (1/\rho)z = (\cancel{u} + a)y \\ \rho a^2 y + \cancel{uz} = (\cancel{u} + a)z \end{cases}$$

$$\boldsymbol{r_2} = \begin{pmatrix} 1 \\ a/\rho \\ a^2 \end{pmatrix}$$

For $\lambda_3 = u - a$,

$$\begin{cases} \cancel{ux} + \rho y = (\cancel{u} - a)x \\ \cancel{uy} + (1/\rho)z = (\cancel{u} - a)y \\ \rho a^2 y + \cancel{uz} = (\cancel{u} - a)z \end{cases}$$

$$\boldsymbol{r_3} = \begin{pmatrix} 1 \\ -a/\rho \\ a^2 \end{pmatrix}$$

Left eigenvector: $\boldsymbol{l}A = \lambda \boldsymbol{l}$.

$$\begin{pmatrix} . & . & . \end{pmatrix} \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{pmatrix} = \lambda \begin{pmatrix} . & . & . \end{pmatrix}$$

For $\lambda_1 = u$,

$$\begin{cases} ux = ux \\ \rho x + \cancel{uy} + \rho a^2 z = \cancel{uy} \\ (1/\rho)y + \cancel{uz} = \cancel{uz} \end{cases}$$

$$\boldsymbol{l_1} = \begin{pmatrix} 1 & 0 & -1/a^2 \end{pmatrix}$$

For $\lambda_2 = u + a$,

$$\begin{cases} \cancel{ux} = (\cancel{u} + a)x \\ \rho x + \cancel{uy} + \rho a^2 z = (\cancel{u} + a)y \\ (1/\rho)y + \cancel{uz} = (\cancel{u} + a)z \end{cases}$$

$$\boldsymbol{l_2} = \begin{pmatrix} 0 & 1 & 1/(\rho a) \end{pmatrix}$$

For $\lambda_3 = u - a$,

$$\begin{cases} \cancel{ux} = (\cancel{u} - a)x \\ \rho x + \cancel{uy} + \rho a^2 z = (\cancel{u} - a)y \\ (1/\rho)y + \cancel{uz} = (\cancel{u} - a)z \end{cases}$$

$$\boldsymbol{l_3} = \begin{pmatrix} 0 & 1 & -1/(\rho a) \end{pmatrix}$$

(e) Consider left rarefaction:

Characteristic: $x = (u - a)t$

So, using characteristic line and invariance,

$$\begin{cases} a = u - \frac{x}{t} \\ u + \frac{2a}{\gamma - 1} = u_L + \frac{2a_L}{\gamma - 1} \end{cases}$$

$$\begin{aligned} u &= u_L + \frac{2a_L}{\gamma - 1} - \frac{2(u - x/t)}{\gamma - 1} \\ u + \frac{2u}{\gamma - 1} &= u_L + \frac{2a_L}{\gamma - 1} + 2\frac{x/t}{\gamma - 1} \\ \frac{\gamma u + u}{\gamma - 1} &= u_L + \frac{2a_L}{\gamma - 1} + 2\frac{x/t}{\gamma - 1} \\ u &= \frac{1}{\gamma + 1}\left((\gamma - 1)u_L + 2a_L + \frac{2x}{t}\right) \\ &= \frac{2}{\gamma + 1}\left(a_L + \frac{\gamma - 1}{2}u_L + \frac{x}{t}\right) \end{aligned}$$

Then,

$$a = \frac{2}{\gamma + 1}\left(a_L + \frac{\gamma - 1}{2}u_L + \frac{x}{t}\right) - \frac{x}{t}$$

(f) **MUSCL-Hancock**.

- Perform slope-limiting, going from piece-wise constant to piece-wise linear approximation. Data reconstruction to define cell boundary values through linear extrapolation and slope limiters.

$$\bar{\boldsymbol{u}}_i^L = \boldsymbol{u}_i - \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i$$

$$\bar{\boldsymbol{u}}_i^R = \boldsymbol{u}_i + \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i$$

where r is the slope ratio and $\boldsymbol{\Delta}_i = \frac{1}{2}(1+\omega)\boldsymbol{\Delta}_{i-1/2}+(1-\omega)\boldsymbol{\Delta}_{i+1/2}$ with $\boldsymbol{\Delta}_{i-1/2} = \boldsymbol{u}_i^n - \boldsymbol{u}_{i-1}^n$ and $\boldsymbol{\Delta}_{i+1/2} = \boldsymbol{u}_{i+1}^n - \boldsymbol{u}_i^n$

- Perform half-time step update of the boundary reconstructed variable.

$$\bar{\boldsymbol{u}}_i^{L,n+1/2} = \bar{\boldsymbol{u}}_i^L - \frac{1}{2}\frac{\Delta t}{\Delta x}(f(\bar{\boldsymbol{u}}_i^R) - f(\bar{\boldsymbol{u}}_i^L))$$

$$\bar{\boldsymbol{u}}_i^{R,n+1/2} = \bar{\boldsymbol{u}}_i^R - \frac{1}{2}\frac{\Delta t}{\Delta x}(f(\bar{\boldsymbol{u}}_i^R) - f(\bar{\boldsymbol{u}}_i^L))$$

- Then, use the resulting states in the HLLC solver (approximate solver) to obtain the fluxes and the update.

  At the $i + 1/2$ interface,

$$\boldsymbol{u}_L = \bar{\boldsymbol{u}}_i^{R,n+1/2} \quad \text{and} \quad \boldsymbol{u}_R = \bar{\boldsymbol{u}}_{i+1}^{L,n+1/2}$$

$$\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \frac{\Delta t}{\Delta x}(\boldsymbol{f}_{i+1/2}^{HLLC} - \boldsymbol{f}_{i-1/2}^{HLLC})$$

  where,

$$\boldsymbol{f}_{i+1/2}^{HLLC} = \begin{cases} f(\boldsymbol{u}_L) & ,0 \leq S_L \\ f(\boldsymbol{u}_L) + S_L(\boldsymbol{u}_L^{HLLC} - \boldsymbol{u}_L) & ,S_L < 0 \leq S^* \\ f(\boldsymbol{u}_R) + S_R(\boldsymbol{u}_R^{HLLC} - \boldsymbol{u}_R) & ,S^* < 0 \leq S_R \\ f(\boldsymbol{u}_R) & ,0 > S_R \end{cases}$$

  Could also use exact Riemann solvers like Godunov method:

$$\boldsymbol{u}_L = \bar{\boldsymbol{u}}_i^{R,n+1/2} \quad \text{and} \quad \boldsymbol{u}_R = \bar{\boldsymbol{u}}_{i+1}^{L,n+1/2}$$

$$\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \frac{\Delta t}{\Delta x}(f(\boldsymbol{u}_{i+1/2}) - f(\boldsymbol{u}_{i-1/2}))$$

  $\boldsymbol{u}_{i+1/2}$ is found through an exact Riemann solver using the half time-step updated boundary states as the left and right states.

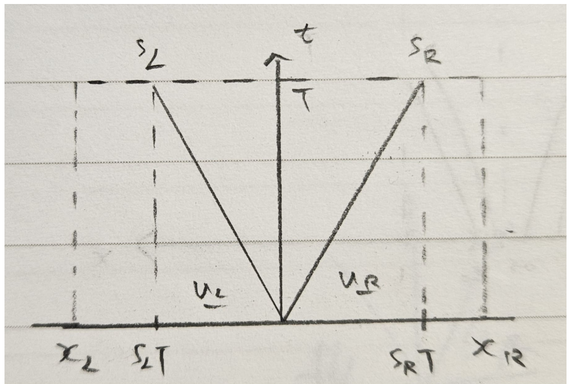
- TVD is ensured through the use of limiters.

(g) HLLC is more diffusive, while MUSCL-Hancock gives steeper waves.
The limiter makes the scheme TVD and gets rid of 2nd order oscillations.

(h) Most diffusive to least diffusive: Minbee, Vanleer, Superbee.

2.3 Q3

3. (a)



(b) A 3-wave diagram includes a contact discontinuity. We can have RCR, RCS, SCR, SCS. 4 combinations of waves.

(c)

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x} = 0$$

(d) Integrate with respect to x and t in the control volume,

$$\int_{x_L}^{x_R} \int_0^T \frac{\partial \boldsymbol{U}(x,t)}{\partial t}\, dt\, dx + \int_{x_L}^{x_R} \int_0^T \frac{\partial \boldsymbol{F}(\boldsymbol{U}(x,t))}{\partial x}\, dt\, dx = 0$$

$$\int_{x_L}^{x_R} \boldsymbol{U}(x,T) - \boldsymbol{U}(x,0)\, dx + \int_0^T \boldsymbol{F}(\boldsymbol{U}(x_R,t)) - \boldsymbol{F}(\boldsymbol{U}(x_L,t))\, dt = 0$$

$$\int_{x_L}^{x_R} \boldsymbol{U}(x,T)\, dx = \int_{x_L}^{x_R} \boldsymbol{U}(x,0)\, dx - T[\boldsymbol{F}(\boldsymbol{U}_R) - \boldsymbol{F}(\boldsymbol{U}_L)]$$

$$\int_{x_L}^{x_R} \boldsymbol{U}(x,T)\, dx = x_R \boldsymbol{U}_R - x_L \boldsymbol{U}_L + T[\boldsymbol{F}(\boldsymbol{U}_L) - \boldsymbol{F}(\boldsymbol{U}_R))]$$

Note that the integrands at assumed piece-wise constant in space and time:

$$\int_0^T \boldsymbol{F}(\boldsymbol{U}(x_R,t))\, dt = T\boldsymbol{F}(\boldsymbol{U}_R) \quad \text{and} \quad \int_{x_L}^{x_R} \boldsymbol{U}(x,0)\, dx = x_R \boldsymbol{U}_R - x_L \boldsymbol{U}_L$$

(e) Three regions are $[x_L, s_L T]$, $[s_L T, s_R T]$ and $[s_R T, x_R]$.

$$\begin{aligned}\int_{x_L}^{x_R} \boldsymbol{U}(x,T)\, dx &= \int_{x_L}^{S_L T} \boldsymbol{U}(x,T)\, dx + \int_{S_L T}^{S_R T} \boldsymbol{U}(x,T)\, dx + \int_{S_R T}^{x_R} \boldsymbol{U}(x,T)\, dx \\ &= s_L T \boldsymbol{U}_L - x_L \boldsymbol{U}_L + \int_{S_L T}^{S_R T} \boldsymbol{U}(x,T)\, dx + x_R \boldsymbol{U}_R - S_R T \boldsymbol{U}_R\end{aligned}$$

Then,

$$\cancel{x_R \boldsymbol{U}_R - x_L \boldsymbol{U}_L} + T[S_L \boldsymbol{U}_L - S_R \boldsymbol{U}_R] + \int_{S_L T}^{S_R T} \boldsymbol{U}(x,T)\, dx = \cancel{x_R \boldsymbol{U}_R - x_L \boldsymbol{U}_L} + T[\boldsymbol{F}(\boldsymbol{U}_L) - \boldsymbol{F}(\boldsymbol{U}_R))]$$

$$\frac{1}{T}\int_{S_L T}^{S_R T} \boldsymbol{U}(x,T)\,dx = \boldsymbol{F}_L - \boldsymbol{F}_R - S_L\boldsymbol{U}_L + S_R\boldsymbol{U}_R$$

$$\frac{1}{T(S_R - S_L)}\int_{S_L T}^{S_R T} \boldsymbol{U}(x,T)\,dx = \frac{S_R\boldsymbol{U}_R - S_L\boldsymbol{U}_L + \boldsymbol{F}_L - \boldsymbol{F}_R}{S_R - S_L}$$

(f) Consider the control volume: $[TS_L, 0] \times [0, T]$:

$$\int_0^T \int_{S_L T}^0 \frac{\partial \boldsymbol{U}(x,t)}{\partial t}\,dx\,dt + \int_0^T \int_{S_L T}^0 \frac{\partial \boldsymbol{F}(\boldsymbol{U}(x,t))}{\partial x}\,dx\,dt = 0$$

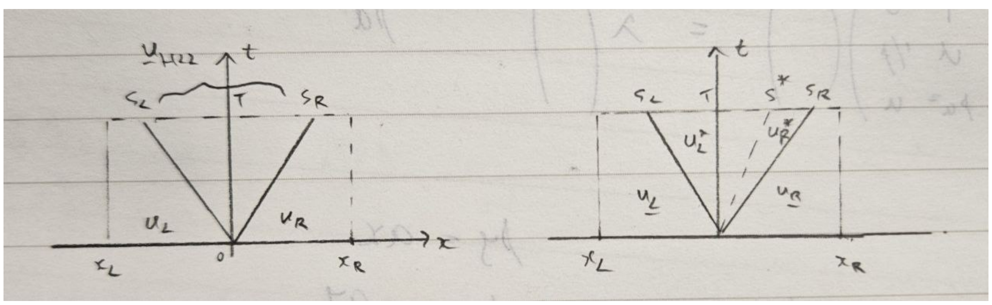
$$\int_{S_L T}^0 \boldsymbol{U}(x,T) - \boldsymbol{U}(x,0)\,dx + \int_0^T \boldsymbol{F}(\boldsymbol{U}(0,t)) - \boldsymbol{F}(\boldsymbol{U}(TS_L,t))\,dt = 0$$

$$\int_{TS_L}^0 \boldsymbol{U}(x,T) + S_L T\boldsymbol{U}_L + T\boldsymbol{F}_{0L} - T\boldsymbol{F}_L = 0$$

$$\boldsymbol{F}_{0L} = \boldsymbol{F}_L - S_L\boldsymbol{U}_L - \frac{1}{T}\int_{TS_L}^0 \boldsymbol{U}(x,T)\,dx$$

(g) We are deriving the HLL scheme.

(h) HLL does not include contact discontinuity but HLLC does.



(i) Slope limiting: going from piece-wise constant to piece-wise linear approximation.

- Boundary values obtained via extrapolation and limiters.
- Perform half timestep evolution.
- Use the resulting states in the HLLC solver to obtain a solution.

Limiters ensure TVD.

3 2019-20 (EXAM)

3.1 Q1

1. (Part I)

   (a)

$$u_i^{n+1} = u_i^n + \frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t^2}{2} + \mathcal{O}(\Delta t^3)$$
$$u_{i+1}^n = u_i^n + \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} + \mathcal{O}(\Delta x^3)$$
$$u_{i-1}^n = u_i^n - \frac{\partial u}{\partial x}\Delta x + \frac{\partial^2 u}{\partial x^2}\frac{\Delta x^2}{2} + \mathcal{O}(\Delta x^3)$$
$$u_i^n = u_i^n$$

   (b) Keep terms up to first order in Δx and Δt,

$$u_t + au_x = 0$$
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) = 0$$
$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$
$$u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) - \frac{1}{2}c(u_{i+1}^n - u_{i-1}^n)$$
$$= \frac{1}{2}(1+c)u_{i-1}^n + \frac{1}{2}(1-c)u_{i+1}^n$$

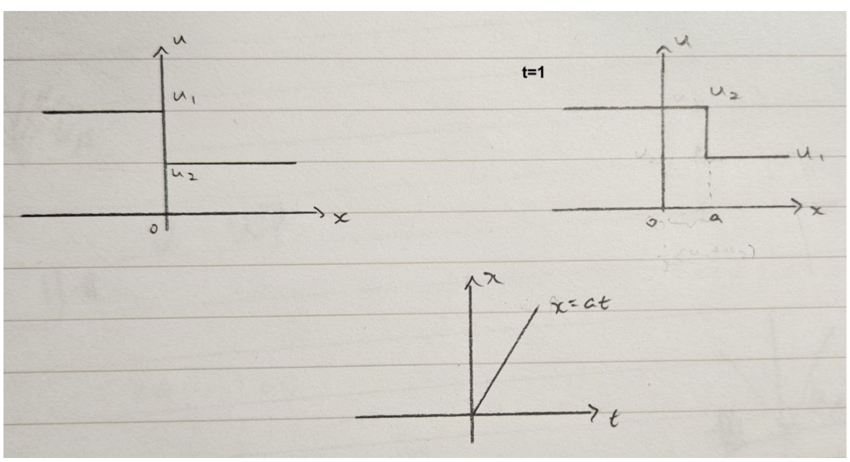
   where $c = \frac{a\Delta t}{\Delta x}$ is the Courant number. It is the Lax-Fridrich scheme.

   (c) Lax-Friedrichs is first-order while Lax-Wendroff and Warming-Beam is second-order.

   (d) For smooth profile, LF is diffusive and LW is dispersive. For sharp discontinuity, LW gives oscillations around the discontinuity, but LF smears the discontinuity.

   (Part II)

   (a)



(b) i. $u(0,t)$ and $u(x, \frac{x}{a})$ are along the characteristic line so they are constants. (Solution is constant along characteristic line)

$$u(0,t) = u_0 = \text{constant} \quad \text{and} \quad u\left(x, \frac{x}{a}\right) = u_1 = \text{constant}$$

ii.

$$u_t + f(u)_x = 0$$

The integral over region A is given by the following:

$$\int_0^{\Delta t} \int_0^{at} dx\, dt \quad \text{or} \quad \int_0^{a\Delta t} \int_{x/a}^{\Delta t} dt\, dx$$

Hence,

$$\int_0^{a\Delta t} \int_{x/a}^{\Delta t} \frac{\partial u}{\partial t}\, dt\, dx + \int_0^{\Delta t} \int_0^{at} \frac{\partial f(u)}{\partial x}\, dx\, dt = 0$$

$$\int_0^{a\Delta t} u(x, \Delta t) - u(x, x/a)\, dx + \int_0^{\Delta t} f(u(at, t)) - f(u(0,t)) = 0$$

$$\int_0^{a\Delta t} u(x, \Delta t) - u_1\, dx + \int_0^{\Delta t} f(u_1) - f(u_0) = 0$$

$$f(u_0) - f(u_1) = \frac{1}{\Delta t} \int_0^{a\Delta t} u(x, \Delta t) - u_1\, dx$$

(c) Monotonicity preserving: if $\{u_i^n\}$ is monotone increasing so is $\{u_i^{n+1}\}$ and if $\{u_i^n\}$ is monotone decreasing so is $\{u_i^{n+1}\}$.

If initial conditions are monotone increasing $u_R > u_L$, then the solution is monotone increasing for all time, vice versa.

(d) Monotone increasing: $u_1 \geq u(x,t)\, \forall x < at$. This means, $f(u_0) - f(u_1) \leq 0$.

3.2 Q2

2. (Part I)

(a) Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 = 16$$

$$\lambda = 5, -3$$

Right eigenvectors: $A\boldsymbol{r} = \lambda \boldsymbol{r}$,

$$\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} a + 4b = \lambda a \\ 4a + b = \lambda b \end{cases}$$

For $\lambda_1 = 5$,

$$\boldsymbol{r}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_1 = -3$,

$$\boldsymbol{r}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b)

$$\Lambda = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$$

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = Q^T$$

Verify:

$$\begin{aligned} A &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 5 \\ -3 & 3 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & 8 \\ 8 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \end{aligned}$$

The columns of Q are the right eigenvectors and the rows of Q^T are the left eigenvectors.

(c)

$$\boldsymbol{v}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sin x \\ \cos x \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin x + \cos x \\ \sin x - \cos x \end{pmatrix} = \begin{pmatrix} \sin\left(x + \frac{\pi}{4}\right) \\ \sin\left(x - \frac{\pi}{4}\right) \end{pmatrix}$$

$$\boldsymbol{W}_t + A\boldsymbol{W}_x = 0$$
$$\boldsymbol{W}_t + Q\Lambda Q^T\boldsymbol{W}_x = 0$$

Matrix Q is orthogonal i.e. $Q^TQ = 1$, meaning its transpose and inverse are the same.

$$Q^T\boldsymbol{W}_t + \Lambda Q^T\boldsymbol{W}_x = 0$$

$$(Q^T\boldsymbol{W})_t + \Lambda(Q^T\boldsymbol{W})_x = 0$$
$$\boldsymbol{v}_t + \Lambda\boldsymbol{v}_x = 0$$

Decoupled:

$$\begin{cases} \frac{\partial v_1}{\partial t} + 5\frac{\partial v_1}{\partial x} = 0 \\ v_1(x,0) = \sin\left(x + \frac{\pi}{4}\right) \end{cases}$$

and

$$\begin{cases} \frac{\partial v_2}{\partial t} - 3\frac{\partial v_2}{\partial x} = 0 \\ v_2(x,0) = \sin\left(x - \frac{\pi}{4}\right) \end{cases}$$

(Part II)

(a) The system is in the form:

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = 0$$

Integrating over control volume to include weak solution,

$$\int_{x-1/2}^{x+1/2}\int_0^{\Delta t} \boldsymbol{U}_t\,dt\,dx + \int_0^{\Delta t}\int_{x-1/2}^{x+1/2} \boldsymbol{F}(\boldsymbol{U})_x\,dx\,dt = 0$$

$$\int_{x-1/2}^{x+1/2} \boldsymbol{U}(x,\Delta t) - \boldsymbol{U}(x,0)\,dx + \int_0^{\Delta t} \boldsymbol{F}(\boldsymbol{U}(x+1/2,t) - \boldsymbol{F}(\boldsymbol{U}(x-1/2,t)\,dt = 0$$

obtain the integral average and let $x - 1/2$ be $x_{i-1/2}$, $x + 1/2$ be $x_{i+1/2}$ and Δt be t^{n+1}.

$$\Delta x\boldsymbol{u}_i^{n+1} - \Delta x\boldsymbol{u}_i^n + \Delta t\boldsymbol{f}_{i+1/2} - \Delta t\boldsymbol{f}_{i-1/2} = 0$$

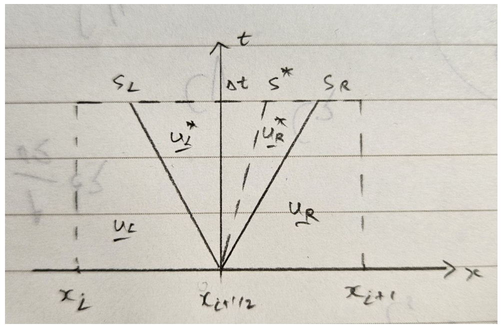
$$\boldsymbol{u}_i^{n+1} = \boldsymbol{u}_i^n - \frac{\Delta t}{\Delta x}(\boldsymbol{f}_{i+1/2} - \boldsymbol{f}_{i-1/2})$$

(b) A Riemann problem is an initial value problem, consisting of a conservation equation and a piece-wise constant initial state with a single discontinuity at $x = x_0$.

$$u_t + f(u)_x = 0 \quad , \quad u(x,0) = \begin{cases} u_L & ,x \le x_0 \\ u_R & ,x \ge x_0 \end{cases}$$

A non-trivial exact solution can be found, so it acts as excellent test cases for numerical methods.

(c) HLLC consists of three sharp waves including a contact discontinuity.



$$\boldsymbol{w}_L = (\rho_L, v_L, p_L) \quad , \quad \boldsymbol{w}_L^* = (\rho_L^*, v^*, p^*)$$
$$\boldsymbol{w}_R^* = (\rho_R^*, v^*, p^*) \quad , \quad \boldsymbol{w}_R = (\rho_R, v_R, p_R)$$

(d) FORCE scheme.

$$\boldsymbol{f}_{i+1/2}^{\text{FORCE}} = \frac{1}{2}(\boldsymbol{f}_{i+1/2}^{RI} + \boldsymbol{f}_{i+1/2}^{LF})$$

(e) In 2D

$$\begin{pmatrix} \rho \\ \rho u_x \\ \rho u_y \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u_x \\ \rho u_x^2 + p \\ \rho u_x u_y \\ (\rho E + p) u_x \end{pmatrix}_x + \begin{pmatrix} \rho u_y \\ \rho u_y u_x \\ \rho u_y^2 + p \\ (\rho E + p) u_y \end{pmatrix}_y = \mathbf{0}$$

(f) **Unsplit method** - accounts for all flux contributions in a single time step:

$$u_{i,j}^{n+1} = u_{i,j}^n - \frac{\Delta t}{\Delta x}(f_{i+1/2,j}^n - f_{i-1/2,j}^n) - \frac{\Delta t}{\Delta y}(g_{i,j+1/2}^n - g_{i,j-1/2}^n)$$

Stability test reveal that:

$$C_x + C_y \leq 1$$

where C_x and C_y are Courant numbers in the x and y directions. We run into CFL halving problems where the stability range is at least halved compared to 1D.

Dimensional splitting - update each direction separately. For example, we can update the x-direction first then the y-direction.

$$\boldsymbol{u}_{i,j}^{n+1} = \mathcal{Y}^{(\Delta t)} \mathcal{X}^{(\Delta t)} \boldsymbol{u}_{i,j}^n \quad \text{(first-order)}$$

Get intermediate state after applying operator $\mathcal{X}^{(\Delta t)}$:

$$\bar{u}_{i,j} = u_{i,j}^n - \frac{\Delta t}{\Delta x}(f_{i+1/2,j}^n - f_{i-1/2,j}^n)$$

Then apply operator $\mathcal{Y}^{(\Delta t)}$ on the intermediate state:

$$u_{i,j}^{n+1} = \bar{u}_{i,j} - \frac{\Delta t}{\Delta x}(g_{i,j+1/2}^n - g_{i,j-1/2}^n)$$

Second-order accurate dimensional splitting can be obtained using Strang splitting.

We compute the stable time step by taking into account the maximum wave speeds in both directions:

$$\Delta t = C \min\left(\frac{\Delta x}{a_{max,x}}, \frac{\Delta y}{a_{max,y}}\right)$$

(g) Reflective boundary conditions has to be used for all boundaries. For N cells in each direction, HLLC uses 1 ghost cell at each boundary (0 and N+1). Reflective boundary is set as

$$u_x[0] = -u_x[1] \quad \text{and} \quad u_y[0] = -u_y[1]$$

$$u_x[N+1] = -u_x[N] \quad \text{and} \quad u_y[N+1] = -u_y[N]$$

3.3 Q3

3. (a) For smooth function $u = u(x)$:

$$TV(u) = \int_{-\infty}^{\infty} |u'(x)| \, dx$$

For discrete function $u^n = \{u_i^n\}$:

$$TV(u^n) = \sum_{i=-\infty}^{\infty} |u_{i+1}^n - u_i^n|$$

(b) The absolute gradient is 4 throughout the domain. Hence, TV(u) = 20 OR use maxima and minima, TV(u) = $|4-5| + |5-0| + |0-5| + |5-0| + |0-4| = 20$.

(c) For (a), the total variation of approximate solution is less than the exact solution. For (b), the total variation of approximate solution is more than the exact solution.

(d) A scheme is TVD if:

$$TV(u^{n+1}) \leq TV(u^n) \, \forall n$$

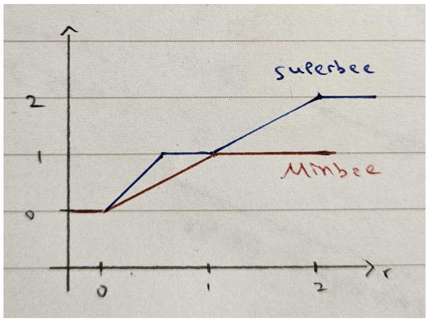
(e) Flux limiting: non-linear combination of fluxes from high-order numerical method and first-order monotone method to obtain a high resolution TVD method, which is higher-order everywhere except near discontinuities where it reverts to the first-order method.

$$f_{i+1/2}^n = f_{i+1/2}^{LO} + \phi_{i+1/2} \left(f_{i+1/2}^{HI} - f_{i+1/2}^{LO} \right)$$

where $\phi_{i+1/2}$ is the limiter function.

(f) FLIC. Uses FORCE (first-order) as f^{LO} and Richtmyer (second-order) as the f^{HI}. It is TVD through the use of limiter function.

(g) Superbee is more compressive than Minbee. Superbee resolves discontinuities better than Minbee, while Minbee allows for more diffusion. However, Superbee might result in squaring of smooth features (suffer small oscillations).



(h) MUSCL-Hancock: slope limiting (linear reconstruction of boundary values + half-time step update) with Riemann solution (exact or approximate).

Conservative MUSCL-Hancock:

- Use the form:
$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = 0$$
- 3 steps: data reconstruction, half-time step update and Riemann solution.
  i. Data reconstruction:
$$\bar{\boldsymbol{u}}_i^L = \boldsymbol{u}_i^n - \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i \quad \text{and} \quad \bar{\boldsymbol{u}}_i^R = \boldsymbol{u}_i^n + \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i$$
  where ξ is the slope limiter and $\boldsymbol{\Delta}_i = \frac{1}{2}(1+\omega)\boldsymbol{\Delta}_{i-1/2} + \frac{1}{2}(1-\omega)\boldsymbol{\Delta}_{i+1/2}$.
  ii. Half time-step update:
$$\boldsymbol{u}_i^{L,n+1/2} = \bar{\boldsymbol{u}}_i^L - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\boldsymbol{f}(\bar{\boldsymbol{u}}_i^R) - \boldsymbol{f}(\bar{\boldsymbol{u}}_i^L)\right)$$
$$\boldsymbol{u}_i^{R,n+1/2} = \bar{\boldsymbol{u}}_i^R - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\boldsymbol{f}(\bar{\boldsymbol{u}}_i^R) - \boldsymbol{f}(\bar{\boldsymbol{u}}_i^L)\right)$$
  iii. Put the updated states in exact or approximate Riemann solver.

Primitive MUSCL-Hancock:

- Use the form:
$$\boldsymbol{W}_t + A\boldsymbol{W}_x = 0$$
- 3 steps: data reconstruction, half-time step update and Riemann solution.
  i. Data reconstruction:
$$\bar{\boldsymbol{w}}_i^L = \boldsymbol{w}_i^n - \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i \quad \text{and} \quad \bar{\boldsymbol{w}}_i^R = \boldsymbol{w}_i^n + \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i$$
  where ξ is the slope limiter and $\boldsymbol{\Delta}_i = \frac{1}{2}(1+\omega)\boldsymbol{\Delta}_{i-1/2} + \frac{1}{2}(1-\omega)\boldsymbol{\Delta}_{i+1/2}$.
  ii. Half time-step update:
$$\boldsymbol{w}_i^{L,n+1/2} = \bar{\boldsymbol{w}}_i^L - \frac{1}{2}\frac{\Delta t}{\Delta x}A(\boldsymbol{w}_i)\bar{\boldsymbol{\Delta}}$$
$$\boldsymbol{w}_i^{R,n+1/2} = \bar{\boldsymbol{w}}_i^R - \frac{1}{2}\frac{\Delta t}{\Delta x}A(\boldsymbol{w}_i)\bar{\boldsymbol{\Delta}}$$
  where $\frac{\partial \boldsymbol{w}}{\partial x} = \frac{\bar{\boldsymbol{\Delta}}}{\Delta x}$
  iii. Put the updated states in exact or approximate Riemann solver.

(i) i.

$$\begin{aligned}
u_{i+1}^{n+1} - u_i^{n+1} &= \left[u_{i+1}^n - C_{i+1/2}\Delta u_{i+1/2} + D_{i+3/2}\Delta u_{i+3/2}\right] - \left[u_i^n - C_{i-1/2}\Delta u_{i-1/2} + D_{i+1/2}\Delta u_{i+1/2}\right] \\
&= (u_{i+1}^n - u_i^n) - C_{i+1/2}(u_{i+1}^n - u_i^n) + D_{i+3/2}(u_{i+2}^n - u_{i+1}^n) \\
&\quad + C_{i-1/2}(u_i^n - u_{i-1}^n) - D_{i+1/2}(u_{i+1}^n - u_i^n) \\
&= (1 - C_{i+1/2} - D_{i+1/2})(u_{i+1}^n - u_i^n) + C_{i-1/2}(u_i^n - u_{i-1}^n) + D_{i+3/2}(u_{i+2}^n - u_{i+1}^n)
\end{aligned}$$

Taking absolute value gives the required expression.

ii. It becomes:

$$
\begin{aligned}
|u_{i+1}^{n+1} - u_i^{n+1}| \leq |u_{i+1}^n - u_i^n| + C_{i-1/2}|u_i^n - u_{i-1}^n| - C_{i+1/2}|u_{i+1}^n - u_i^n| \\
+ D_{i+3/2}|u_{i+2}^n - u_{i+1}^n| - D_{i+1/2}|u_{i+1}^n - u_i^n|
\end{aligned}
$$

iii. Summing over i on the LHS will give us $TV(u^{n+1})$:

$$
\begin{aligned}
TV(u^{n+1}) \leq \sum_i |u_{i+1}^n - u_i^n| + \sum_i C_{i-1/2}|u_i^n - u_{i-1}^n| - \sum_i C_{i+1/2}|u_{i+1}^n - u_i^n| \\
+ \sum_i D_{i+3/2}|u_{i+2}^n - u_{i+1}^n| - \sum_i D_{i+1/2}|u_{i+1}^n - u_i^n|
\end{aligned}
$$

Replacing i with j+1 on selected terms allow us to spot cancellations of summation:

$$
\begin{aligned}
TV(u^{n+1}) \leq TV(u^n) + \cancel{\sum_j C_{j+1/2}|u_{j+1}^n - u_j^n| - \sum_i C_{i+1/2}|u_{i+1}^n - u_i^n|} \\
\cancel{+ \sum_i D_{i+3/2}|u_{i+2}^n - u_{i+1}^n| - \sum_j D_{j+3/2}|u_{j+2}^n - u_{j+1}^n|}
\end{aligned}
$$

Hence,

$$
TV(u^{n+1}) \leq TV(u^n)
$$

4 2020-21 (EXAM)

4.1 Q1

1. Same question as 2023-24 (Mock) Q1. Link

4.2 Q2

2. Same question as 2022-23 (Mock) Q2. Link

4.3 Q3

3. (a) 1D conservation law for $\boldsymbol{Q}$:

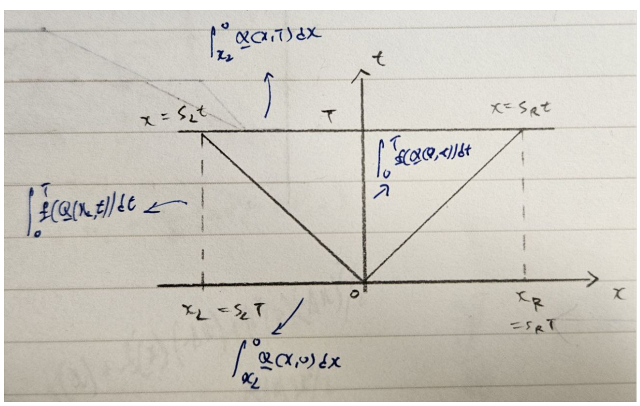
$$\boldsymbol{Q}_t + \boldsymbol{F}(\boldsymbol{Q})_x = 0$$

Integrating over $[x_L, 0] \times [0, T]$,

$$\int_{x_L}^{0} \int_{0}^{T} \boldsymbol{Q}_t \, dt \, dx + \int_{0}^{T} \int_{x_L}^{0} \boldsymbol{f}(\boldsymbol{Q})_x \, dx \, dt = 0$$

$$\int_{x_L}^{0} \boldsymbol{Q}(x,T) - \boldsymbol{Q}(x,0) \, dx + \int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(0,t)) - \boldsymbol{f}(\boldsymbol{Q}(x_L,t)) \, dt = 0$$

$$\int_{x_L}^{0} \boldsymbol{Q}(x,T) \, dx = \int_{x_L}^{0} \boldsymbol{Q}(x,0) \, dx + \int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(x_L,t)) \, dt - \int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(0,t)) \, dt$$



(b)

$$\int_{x_L}^{0} \boldsymbol{Q}(x,T) \, dx = (0 - x_L)\boldsymbol{Q}_L + (T-0)\boldsymbol{f}(\boldsymbol{Q}_L) - (T-0)\boldsymbol{F}_0$$

Dividing throughout by T,

$$\boldsymbol{F}_0 = -\frac{x_L}{T}\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \frac{1}{T}\int_{x_L}^{0} \boldsymbol{Q}(x,T) \, dx$$

$$= -S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \frac{1}{T}\int_{x_L}^{0} \boldsymbol{Q}(x,T) \, dx$$

(c) Using right state:

$$\int_0^{x_R} \boldsymbol{Q}(x,T)\,dx = \int_0^{x_R} \boldsymbol{Q}(x,0)\,dx + \int_0^T \boldsymbol{f}(\boldsymbol{Q}(0,t))\,dt - \int_0^T \boldsymbol{f}(\boldsymbol{Q}(x_R,t))\,dt$$

$$T\boldsymbol{F}_0 = \int_0^{x_R} \boldsymbol{Q}(x,T)\,dx + T\boldsymbol{f}(\boldsymbol{Q}_R) - x_R\boldsymbol{Q}_R$$

$$\boldsymbol{F}_0 = -S_R\boldsymbol{Q}_R + \boldsymbol{f}(\boldsymbol{Q}_R) + \frac{1}{T}\int_0^{x_R} \boldsymbol{Q}(x,T)\,dx$$

Hence, equating both expressions for $\boldsymbol{F}_0$,

$$-S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \frac{1}{T}\int_{x_L}^0 \boldsymbol{Q}(x,T)\,dx = -S_R\boldsymbol{Q}_R + \boldsymbol{f}(\boldsymbol{Q}_R) + \frac{1}{T}\int_0^{x_R} \boldsymbol{Q}(x,T)\,dx$$

$$\frac{1}{T}\int_{x_L}^0 \boldsymbol{Q}(x,T)\,dx + \frac{1}{T}\int_0^{x_R} \boldsymbol{Q}(x,T)\,dx = -S_L\boldsymbol{Q}_L + s_R\boldsymbol{Q}_R + \boldsymbol{f}(\boldsymbol{Q}_L) - \boldsymbol{f}(\boldsymbol{Q}_R)$$

$$\int_{x_L}^{x_R} \boldsymbol{Q}(x,T)\,dx = T\left[S_R\boldsymbol{Q}_R - S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \boldsymbol{f}(\boldsymbol{Q}_R)\right]$$

(d) HLL assume intermediate state is constant.

$$(x_R - x_L)\boldsymbol{Q}^{HLL} = T\left[S_R\boldsymbol{Q}_R - S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \boldsymbol{f}(\boldsymbol{Q}_R)\right]$$

$$\boldsymbol{Q}^{HLL} = \frac{S_R\boldsymbol{Q}_R - S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \boldsymbol{f}(\boldsymbol{Q}_R)}{S_R - S_L}$$

(e) $\hat{\boldsymbol{f}}^{HLL}$ is not the same as $\boldsymbol{f}(\boldsymbol{Q}^{HLL})$ because HLL flux is obtained by using the jump condition across the sharp waves i.e.

$$\boldsymbol{f}^{HLL} - \boldsymbol{f}_K = S_K(\boldsymbol{Q}^{HLL} - \boldsymbol{Q}_K) \quad \text{where} \quad K \in [L, R]$$

(f) Full HLL flux at cell interface is:

$$\boldsymbol{f}_{i+1/2}^{HLL} = \begin{cases} \boldsymbol{f}(\boldsymbol{Q}_L) & , 0 \le S_L \\ \hat{\boldsymbol{f}}^{HLL} & , S_L < 0 < S_R \\ \boldsymbol{f}(\boldsymbol{Q}_R) & , S_R \le 0 \end{cases}$$

(g) HLLC includes a third sharp wave which is the contact discontinuity, and it has two intermediate states.

(h) The HLL state $\boldsymbol{Q}^{HLL}$ is now made up of two intermediate states separated by the contact discontinuity. We have,

$$\int_{x_L}^{x_R} \boldsymbol{Q}(x,T)\,dx = \int_{x_L}^{S_*T} \boldsymbol{Q}_{*L}(x,T)\,dx + \int_{S_*T}^{x_R} \boldsymbol{Q}_{*R}(x,T)\,dx$$

$$(S_RT - S_LT)\boldsymbol{Q}^{HLL} = (S_*T - S_LT)\boldsymbol{Q}_{*L} + (S_RT - S_*T)\boldsymbol{Q}_{*R}$$

$$\boldsymbol{Q}^{HLL} = \left(\frac{S_* - S_L}{S_R - S_L}\right)\boldsymbol{Q}_{*L} + \left(\frac{S_R - S_*}{S_R - S_L}\right)\boldsymbol{Q}_{*R}$$

where S_* is the speed of the contact discontinuity and the star states are the solutions to the left and right of the contact discontinuity.

(i) Rankine-Hugoniot conditions:

$$\boldsymbol{f}_1 - \boldsymbol{f}_2 = S(\boldsymbol{Q}_1 - \boldsymbol{Q}_2)$$

For Euler equations:

$$\boldsymbol{Q} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} \quad \text{and} \quad \boldsymbol{f} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}$$

Using the RH conditions for conservation of mass and momentum at the left and right sharp wave to obtain 4 equations:

$$\begin{cases} \rho_L u_L - \rho_{*L} S_* = S_L(\rho_L - \rho_{*L}) \\ \rho_L u_L^2 + p_L - \rho_{*L} S_*^2 - p_* = S_L(\rho_L u_L - \rho_{*L} S_*) \\ \rho_{*R} S_* - \rho_R u_R = S_R(\rho_{*R} - \rho_R) \\ \rho_{*R} S_*^2 + p_* - \rho_R u_R^2 - p_R = S_R(\rho_{*R} S_* - \rho_R u_R) \end{cases}$$

I want to remove p_*.

$$p_* = \rho_L u_L^2 + p_L - \rho_{*L} S_*^2 - S_L^2(\rho_L - \rho_{*L}) = S_R^2(\rho_{*R} - \rho_R) - \rho_{*R} S_*^2 + \rho_R u_R^2 + p_R$$

$$\rho_L u_L^2 + p_L - \rho_{*L}(S_*^2 - S_L^2) - S_L^2 \rho_L = \rho_R u_R^2 + p_R - \rho_{*R}(S_*^2 - S_R^2) - S_R^2 \rho_R \tag{1}$$

I also want to remove ρ_{*L} and ρ_{*R},

$$\rho_{*L}(S_* - S_L) = \rho_L u_L - S_L \rho_L \quad \Rightarrow \quad \rho_{*L} = \frac{\rho_L u_L - S_L \rho_L}{S_* - S_L}$$

$$\rho_{*R}(S_* - S_R) = \rho_R u_R - S_R \rho_R \quad \Rightarrow \quad \rho_{*R} = \frac{\rho_R u_R - S_R \rho_R}{S_* - S_R}$$

Substituting into equation (1),

$$\rho_L u_L^2 + p_L - (\rho_L u_L - S_L \rho_L)(S_* + S_L) - S_L^2 \rho_L = \rho_R u_R^2 + p_R - (\rho_R u_R - S_R \rho_R)(S_* + S_R) - S_R^2 \rho_R$$

Make S_* the subject,

$$S_*[\rho_R(u_R - S_R) - \rho_L(u_L - S_L)] = \rho_R u_R^2 + p_R - \cancel{S_R^2 \rho_R} - \rho_L u_L^2 - p_L + \cancel{S_L^2 \rho_L} - \rho_R(u_R - \cancel{S_R})S_R + \rho_L(u_L - \cancel{S_L})S_L$$

$$S_* = \frac{p_R - p_L + \rho_L u_L(S_L - u_L) - \rho_R u_R(S_R - u_R)}{\rho_R(u_R - S_R) - \rho_L(u_L - S_L)}$$

$$S_* = \frac{p_R - p_L + \rho_L u_L(S_L - u_L) - \rho_R u_R(S_R - u_R)}{\rho_L(S_L - u_L) - \rho_R(S_R - u_R)}$$

(j) Initially, equation (2) has 3 equations for 4 unknowns $\{\rho_{*L}, \rho_{*R}, S_*, p_*\}$. After obtaining an expression for S_* in terms of known left and right states, we have 3 equations and 3 unknowns $\{\rho_{*L}, \rho_{*R}, p_*\}$ so we can solve the equations.

5 2022-23 (MOCK)

5.1 Q1

1. Same question as 2023-24 (Mock) Q1. Link

5.2 Q2

2. (a) The numerical solution suffer from oscillations near discontinuities. It is also dispersive.

(b) Flux limiting can be used to give us a high-resolution total variation diminishing (TVD) method, through using a non-linear combination of a high-order scheme and a low-order monotone scheme. Higher-order methods are used everywhere except near discontinuities, where first-order method is switched on.

$$\boldsymbol{f}_{i+1/2}^{TVD} = \boldsymbol{f}_{i+1/2}^{LO} + \phi_{i+1/2}(\boldsymbol{f}_{i+1/2}^{HI} - \boldsymbol{f}_{i+1/2}^{LO})$$

where $\phi_{i+1/2} = \phi_{i+1/2}(r)$ is the limiter and r is the slope ratio.

(c) We can fit a linear function through $\boldsymbol{u}_i^n$:

$$\frac{\boldsymbol{u}_i(x) - \boldsymbol{u}_i^n}{x - x_i} = \frac{\boldsymbol{\Delta}_i}{\Delta x} \quad \Rightarrow \quad \boldsymbol{u}_i(x) = \boldsymbol{u}_i^n + (x - x_i)\frac{\boldsymbol{\Delta}_i}{\Delta x}$$

where $\boldsymbol{\Delta}_i$ is a measure of slope, given by the combination of neighbouring slopes:

$$\boldsymbol{\Delta}_i = \frac{1}{2}(\boldsymbol{\Delta}_{i-1/2} + \boldsymbol{\Delta}_{i+1/2}) - \frac{1}{2}w(\boldsymbol{\Delta}_{i+1/2} - \boldsymbol{\Delta}_{i-1/2}) \quad , w \in [-1, 1]$$

where $\boldsymbol{\Delta}_{i+1/2} = \boldsymbol{u}_{i+1}^n - \boldsymbol{u}_i^n$ and $\boldsymbol{\Delta}_{i-1/2} = \boldsymbol{u}_i^n - \boldsymbol{u}_{i-1/2}^n$. Then, the left and right boundary states are given by:

$$\boldsymbol{u}_i^L = \boldsymbol{u}_i^n - \frac{1}{2}\boldsymbol{\Delta}_i \quad , \quad \boldsymbol{u}_i^R = \boldsymbol{u}_i^n + \frac{1}{2}\boldsymbol{\Delta}_i$$

(d) Use r and $1/r$: consider the behaviour for two values of slope ratio reciprocal of each other:

$$\hat{r} = \frac{\Delta_{i+1/2}}{\Delta_{i-1/2}} = \frac{\Delta_+}{\Delta_-} \quad \text{and} \quad r = \frac{\Delta_-}{\Delta_+}$$

For Minbee and Superbee, they will introduce antisymmetric behaviour due to the r and 2r.

For Van-Leer,

$$\xi(r) = \min\left(\frac{2r}{1+r}, \frac{2}{1+r}\right) \quad , \quad \xi(\hat{r}) = \min\left(\frac{2\hat{r}}{1+\hat{r}}, \frac{2}{1+\hat{r}}\right)$$

$$\xi(r) = \min\left(\frac{2\Delta_-}{\Delta_+ + \Delta_-}, \frac{2\Delta_+}{\Delta_+ + \Delta_-}\right) \quad , \quad \xi(\hat{r}) = \min\left(\frac{2\Delta_+}{\Delta_+ + \Delta_-}, \frac{2\Delta_-}{\Delta_+ + \Delta_-}\right)$$

Hence, symmetric.

For Van-Albada,

$$\xi(r) = \min\left(\frac{r(1+r)}{1+r^2}, \frac{2}{1+r}\right) \quad , \quad \xi(\hat{r}) = \min\left(\frac{\hat{r}(1+\hat{r})}{1+\hat{r}^2}, \frac{2}{1+\hat{r}}\right)$$

$$\xi(r) = \min\left(\frac{\Delta_-(\Delta_+ + \Delta_-)}{\Delta_+^2 + \Delta_-^2}, \frac{2\Delta_+}{\Delta_+ + \Delta_-}\right) \quad , \quad \xi(\hat{r}) = \min\left(\frac{\Delta_+(\Delta_+ + \Delta_-)}{\Delta_-^2 + \Delta_+^2}, \frac{2\Delta_-}{\Delta_+ + \Delta_-}\right)$$

Hence, Van-Albada will introduce antisymmetric behaviour.

(e) MUSCL-Hancock method

1. Slope limiting is performed on linearly reconstructed slopes:

$$\bar{\boldsymbol{u}}_i^L = \boldsymbol{u}_i^n - \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i \quad \text{and} \quad \bar{\boldsymbol{u}}_i^R = \boldsymbol{u}_i^n + \frac{1}{2}\xi(r)\boldsymbol{\Delta}_i$$

where $\xi(r)$ is a function of slope ratio.

2. Half-time step update of boundary states:

$$\bar{\boldsymbol{u}}_i^{L,n+1/2} = \bar{\boldsymbol{u}}_i^L - \frac{1}{2}\frac{\Delta t}{\Delta x}(\boldsymbol{f}(\bar{\boldsymbol{u}}_i^R) - \boldsymbol{f}(\bar{\boldsymbol{u}}_i^L))$$

$$\bar{\boldsymbol{u}}_i^{R,n+1/2} = \bar{\boldsymbol{u}}_i^R - \frac{1}{2}\frac{\Delta t}{\Delta x}(\boldsymbol{f}(\bar{\boldsymbol{u}}_i^R) - \boldsymbol{f}(\bar{\boldsymbol{u}}_i^L))$$

3. We then use these updated boundary states in a Riemann solver, either exact or approximate, by noting that at the cell boundary $x_{i+1/2}$ we have:

$$\boldsymbol{u}_L = \bar{\boldsymbol{u}}_i^{R,n+1/2} \quad \text{and} \quad \boldsymbol{u}_R = \bar{\boldsymbol{u}}_{i+1}^{L,n+1/2}$$

(f) Dimensional splitting

(g) Performing a single operation has a CFL number halved of the underlying numerical scheme. Also, it is not easy to implement an unsplit numerical scheme which takes into account of waves moving diagonally.

(h) Strang splitting:

$$\boldsymbol{u}_{i,j}^{n+1} = \frac{1}{2}\left[\mathcal{Y}^{\Delta t}\mathcal{X}^{\Delta t} + \mathcal{X}^{\Delta t}\mathcal{Y}^{\Delta t}\right](\boldsymbol{u}^n)$$

$$\boldsymbol{u}_{i,j}^{n+1} = \mathcal{X}^{\frac{1}{2}\Delta t}\mathcal{Y}^{\Delta t}\mathcal{X}^{\frac{1}{2}\Delta t}(\boldsymbol{u}^n)$$

$$\boldsymbol{u}_{i,j}^{n+2} = \frac{1}{2}\left[\mathcal{Y}^{\Delta t}\mathcal{X}^{\Delta t}\mathcal{X}^{\Delta t}\mathcal{Y}^{\Delta t}\right](\boldsymbol{u}^n)$$

5.3 Q3

3. (a) 1D conservation law

$$\partial_t \boldsymbol{Q} + \partial_x \boldsymbol{f}(\boldsymbol{Q}) = 0$$

Integrate both sides within control volume $= [x_L, 0] \times [0, T]$:

$$\int_{x_L}^{0}\int_{0}^{T} \frac{\partial \boldsymbol{Q}}{\partial t} dt dx + \int_{0}^{T}\int_{x_L}^{0} \frac{\partial \boldsymbol{f}(\boldsymbol{Q})}{\partial x} dx dt = 0$$

$$\int_{x_L}^{0} \boldsymbol{Q}(x,T)dx - \int_{x_L}^{0} \boldsymbol{Q}(x,0)dx + \int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(0,t)) - \int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(x_L,t)) = 0$$

$$\int_{x_L}^{0} \boldsymbol{Q}(x,T)dx = \int_{x_L}^{0} \boldsymbol{Q}(x,0)dx - \underbrace{\int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(0,t))}_{\boldsymbol{F}_0} + \int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(x_L,t))$$

(b)

$$T\boldsymbol{F}_0 = \int_{x_L}^{0} \boldsymbol{Q}(x,0)dx + T\boldsymbol{f}(\boldsymbol{Q}_L) - \int_{x_L}^{0} \boldsymbol{Q}(x,T)dx$$

$$= (0 - S_L T)\boldsymbol{Q}_L + T\boldsymbol{f}(\boldsymbol{Q}_L) - \int_{x_L}^{0} \boldsymbol{Q}(x,T)dx$$

$$\boldsymbol{F}_0 = -S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \frac{1}{T}\int_{x_L}^{0} \boldsymbol{Q}(x,T)dx$$

(c) Using right state:

$$\int_{0}^{x_R}\int_{0}^{T} \frac{\partial \boldsymbol{Q}}{\partial t} dt dx + \int_{0}^{T}\int_{0}^{x_R} \frac{\partial \boldsymbol{f}(\boldsymbol{Q})}{\partial x} dx dt = 0$$

$$\int_{0}^{x_R} \boldsymbol{Q}(x,T)dx - \int_{0}^{x_R} \boldsymbol{Q}(x,0)dx + \int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(x_R,t)) - \underbrace{\int_{0}^{T} \boldsymbol{f}(\boldsymbol{Q}(0,t))}_{\boldsymbol{F}_0} = 0$$

$$\boldsymbol{F}_0 = -S_R\boldsymbol{Q}_R + \boldsymbol{f}(\boldsymbol{Q}_R) + \frac{1}{T}\int_{0}^{x_R} \boldsymbol{Q}(x,T)dx$$

Equating:

$$\frac{1}{T}\left(\int_{x_L}^{0} \boldsymbol{Q}(x,T)dx + \int_{0}^{x_R} \boldsymbol{Q}(x,T)dx\right) = -S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) + S_R\boldsymbol{Q}_R - \boldsymbol{f}(\boldsymbol{Q}_R)$$

$$\int_{x_L}^{x_R} \boldsymbol{Q}(x,T)dx = T\left(S_R\boldsymbol{Q}_R - S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \boldsymbol{f}(\boldsymbol{Q}_R)\right)$$

(d) HLL assume intermediate state is constant.

$$\tilde{\boldsymbol{u}}(x,T) = \begin{cases} \boldsymbol{Q}_L & , x < TS_L \\ \boldsymbol{Q}^{HLL} & , TS_L < x < TS_R \\ \boldsymbol{Q}_R & , x > TS_R \end{cases}$$

$$(TS_R - TS_L)\boldsymbol{Q}^{HLL} = T\left(S_R\boldsymbol{Q}_R - S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \boldsymbol{f}(\boldsymbol{Q}_R)\right)$$
$$\boldsymbol{Q}^{HLL} = \frac{S_R\boldsymbol{Q}_R - S_L\boldsymbol{Q}_L + \boldsymbol{f}(\boldsymbol{Q}_L) - \boldsymbol{f}(\boldsymbol{Q}_R)}{S_R - S_L}$$

(e) HLL flux is obtained by using the jump condition across shock wave. $\boldsymbol{f}^{HLL} \neq \boldsymbol{f}(\boldsymbol{Q}^{HLL})$

(f)

$$\boldsymbol{f}^{HLL}_{i+1/2} = \begin{cases} \boldsymbol{f}_L & ,0 \leq S_L \\ \hat{\boldsymbol{f}}^{HLL} & ,S_L < 0 < S_R \\ \boldsymbol{f}_R & ,0 \geq S_R \end{cases}$$

(g) HLLC is a 3-wave approximate Riemann solver, including the contact discontinuity.

(h) S^* is the speed of the contact discontinuity. $\boldsymbol{Q}_{*L}$ and $\boldsymbol{Q}_{*R}$ are the solution to the left and right of the contact.

Refer to 2020-21 (Exam) Q3 Link

(i) ?

(j) ?

6 2022-23 (EXAM)

6.1 Q1

1. Same question as 2023-24 (Mock) Q2. Link

6.2 Q2

2. (a) Integral average for state vector:

$$\boldsymbol{u}_i^n = \frac{1}{\Delta x}\int_{x_i-\frac{1}{2}\Delta x}^{x_i+\frac{1}{2}\Delta x} \boldsymbol{u}(x,t^n)\,dx$$

(b) Integral average for flux:

$$\boldsymbol{f}_{i+1/2}^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \boldsymbol{f}(\boldsymbol{u}(x_{i+1/2},t))\,dt$$

(c)

$$\frac{\boldsymbol{u}_i^{n+1}-\boldsymbol{u}_i^n}{\Delta t} + \frac{\boldsymbol{f}_{i+1/2}^n - \boldsymbol{f}_{i+1/2}^n}{\Delta x} = 0$$

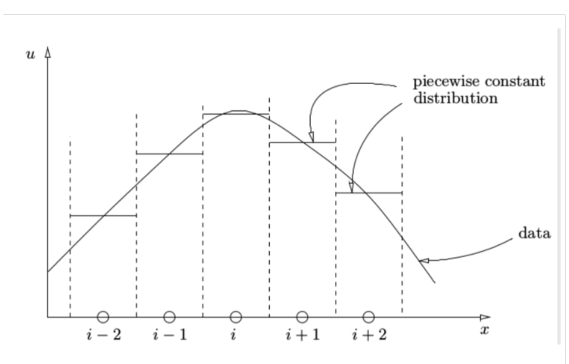
As $\Delta x, \Delta t \to 0$, we recover the conservation law, $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$. When solving the equation with a numerical method, we assume piece-wise constant data within the numerical stencil $[x_i - \frac{1}{2}\Delta x, x_i + \frac{1}{2}\Delta x] \times [t^n, t^{n+1}]$.

- The quantities defined in the previous two questions assume that the solution is known everywhere in the cell at t^n, and the fluxes are known at both boundaries for the full time step.
- In practice, these fluxes cannot be known at the intermediate times between t^n and t^{n+1}, and thus they must be a numerical approximation.

(d) A Riemann problem is an initial value problem consisting of a conservation equation and two piece-wise constant initial state separated by a single discontinuity at $x = x_0$ (draw it). A non-trivial exact solution can be found, acting as excellent validation tests for numerical methods.

(e) **Godunov's method**:

- Godunov's method assumes the integral averages in each cell $\boldsymbol{u}_i^n$ is piece-wise constant.



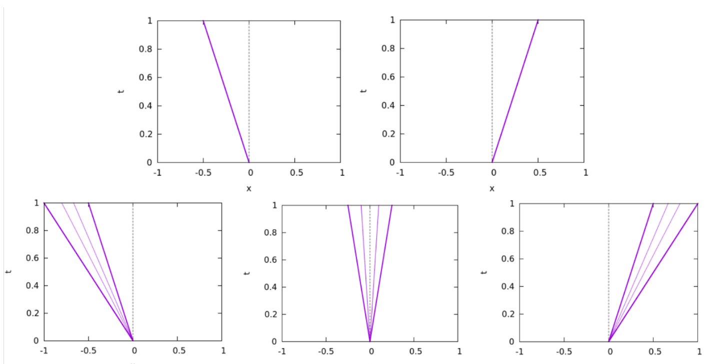
- This sets up a Riemann problem at $x_{i+1/2}$ between neighbouring cells, $\boldsymbol{u}_i^n$ and $\boldsymbol{u}_{i+1}^n$.
- The Riemann problem is self-similar so the solution at $x_{i+1/2}$ is constant for $t > t^n$.
- Solving the Riemann problems at every cell boundary gives us the state at each interface $\boldsymbol{u}_{i+1/2}^n$.
- We obtain the interfacial flux by: $\boldsymbol{f}_{i+1/2}^n = \boldsymbol{f}(\boldsymbol{u}_{i+1/2}^n)$.
- Likewise for cells x_i and x_{i-1} to obtain $\boldsymbol{f}_{i-1/2}^n$.

(f)

$$\text{Burger's equation:} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

(g) When looking at Riemann problems, think about characteristic variable which is u, and $\lambda = u$.

There are 5 possible solutions to the Riemann problem at $x_{i+1/2}$ between u_i and u_{i+1}.



Shock wave form for $\lambda(u_i) > S > \lambda(u_{i+1})$, that is, $u_i^n > u_{i+1}^n$. Rankine-Hugoniot condition gives the shock speed as:

$$S = \frac{u_i^n + u_{i+1}^n}{2}$$

Rarefaction wave forms when $\lambda(u_i) < \lambda(u_{i+1})$, that is, $u_i^n < u_{i+1}^n$. It is bounded by lines with slope $\frac{1}{u_i^n}$ and $\frac{1}{u_{i+1}^n}$ in the x-t diagram, and taking all values in between. For the case where rarefaction covers $x_{i+1/2}$, $u = 0$ is a characteristic along $x =$ constant.

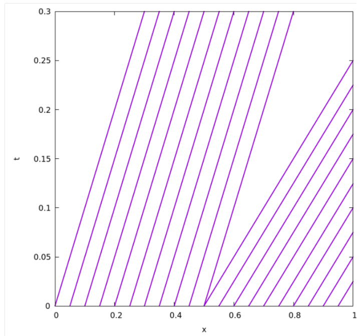
Solving for the interfacial state gives:

$$u_{i+1/2}^n = \begin{cases} u_i^n & , u_i^n > u_{i+1}^n \ , \ S > 0 \\ u_{i+1}^n & , u_i^n > u_{i+1}^n \ , \ S < 0 \\ u_i^n & , u_i^n < u_{i+1}^n \ , \ u_i^n > 0 \\ 0 & , u_i^n < u_{i+1}^n \ , \ u_i^n \leq 0 \leq u_{i+1}^n \\ u_{i+1}^n & , u_i^n < u_{i+1}^n \ , \ u_{i+1}^n < 0 \end{cases}$$

(h) To get unique Riemann problem solution: entropy condition. The entropy condition distinguishes between weak solutions and pick the one that is physically correct. It rejects expansion shock as a physical solution, as it is entropy-violating, while admitting rarefaction waves.

$$\lambda(u_i) > S > \lambda(u_{i+1})$$

(i) Riemann problem with diverging characteristics gives 2 weak solutions: a rarefaction fan and a rarefaction shock. Which one is physically correct i.e stable under perturbation?



- Consider a perturbation of a Riemann problem solution, with three initial states:

$$u(x) = \begin{cases} u_L & , x \leq x_L \\ u_M & , x_L < x < x_R \\ u_R & , x \geq x_R \end{cases}$$

  with interval $\Delta x = x_R - x_L$.
- If we assume the solution is a rarefaction fan, the characteristic diagram is unchanged and we recover the rarefaction wave. However, if we assume the solution is a rarefaction shock, we obtain two distinct waves, even in the limit $\Delta x \to 0$. The rarefaction shock solution is unstable.

$$S_1 = \frac{1}{2}(u_L + u_M) \quad \text{and} \quad S_2 = \frac{1}{2}(u_M + u_R)$$

- Hence, for consistent solutions under perturbation of initial data, we only permit the rarefaction wave.

(j) CFL condition gives the maximum stable timestep:

$$\Delta t = \frac{C\Delta x}{a_{max}}$$

where C is the Courant number dependent on the numerical method and a_{max} is the maximum physical wave speed in the domain,

$$a_{max} = \max_i(|u_i|)$$

The CFL condition asserts that the numerical waves should propagate at least as fast as the physical wave. This means that the numerical wave speed of $\Delta x/\Delta t$ must be at least as fast as the physical wave speed.

$$\frac{\Delta x}{\Delta t} \geq |a_{\max}|$$

This inequality is usually enforced by choosing a Courant number $C = \frac{\Delta t|a_{\max}|}{\Delta x}$ and $0 < C < 1$.

(k) **Why is conservation form important? (Burger's equation)**

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0 \quad \text{or} \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Consider discretising both of the equations using backwards difference:

$$\text{Conservative form: } u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right]$$

$$\text{Primitive form: } u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[(u_i^n)^2 - u_i^n u_{i-1}^n\right]$$

Summing over the domain where $i \in [1, M]$,

For conservative form:

$$\begin{aligned}
\sum_{i=1}^{M} u_i^{n+1} &= \sum_{i=1}^{M} u_i^n - \frac{\Delta t}{\Delta x}\sum_{i=1}^{M}\left[\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right] \\
&= \sum_{i=1}^{M} u_i^n - \frac{\Delta t}{\Delta x}\left[\cancel{\frac{1}{2}(u_1^n)^2} - \frac{1}{2}(u_0^n)^2 + \cancel{\frac{1}{2}(u_2^n)^2} - \cancel{\frac{1}{2}(u_1^n)^2} + \dots + \frac{1}{2}(u_M^n)^2 - \cancel{\frac{1}{2}(u_{M-1}^n)^2}\right] \\
&= \sum_{i=1}^{M} u_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(u_M^n)^2 - \frac{1}{2}(u_0^n)^2\right] \\
&= \sum_{i=1}^{M} u_i^n - \frac{\Delta t}{\Delta x}\left[f_{M+1/2}(u) - f_{-1/2}(u)\right]
\end{aligned}$$

The total change of u depends only on the contribution from the boundaries, and this always holds in the presence of a discontinuity. So, the discontinuity will be correctly positioned.

For non-conservative form:

$$\sum_{i=1}^{M} u_i^{n+1} = \sum_{i=1}^{M} u_i^n - \frac{\Delta t}{\Delta x}\sum_{i=1}^{M}\left[(u_i^n)^2 - u_i^n u_{i-1}^n\right]$$

The terms only cancel in the limit $\Delta x \to 0$ where we hope to have $u_i^n = u_{i-1}^n$. However, if we have a discontinuity between i and $i+1$, the terms will not cancel even in the limit of $\Delta x \to 0$,

$$u_i^{n+1} + u_{i+1}^{n+1} = u_i^n + u_{i+1}^n - \frac{\Delta t}{\Delta x}[(u_{i+1}^2) - \underbrace{u_{i+1}^n u_i^2 + (u_i^n)^2}_{\text{never cancel}} - u_i^n u_{i-1}^n]$$

This results in a gain or loss of u when it updates, meaning the discontinuity is guaranteed to be incorrectly placed. **Non-conservative methods can be guaranteed to get the position of a discontinuity wrong.** Conservative form are shock capturing methods.

6.3 Q3

3. Same question as 2023-24 (Mock) Q3. Link

7 2023-24 (MOCK)

7.1 Q1

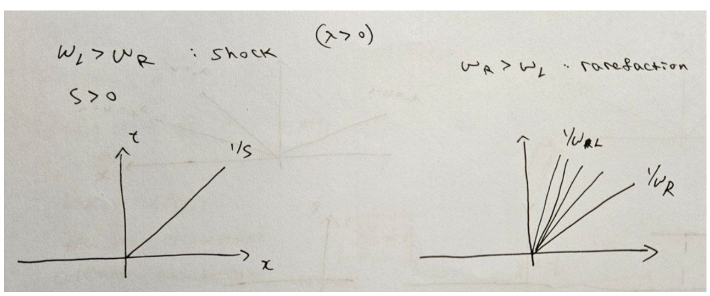
1. (Part I)

   (a)
   $$\partial_t w + \partial_x f(w) = 0 = \partial_t w + \partial_w f(w)\partial_x w$$
   The eigenvalue is
   $$\lambda(w) = \frac{\partial f(w)}{\partial w} = w^{1/2} > 0$$
   If $w > 0$, the eigenvalue is real so the PDE is hyperbolic. If $w < 0$, the eigenvalue is imaginary, so the PDE is elliptic.

   (b) The characteristic variable is w.

   (c) RH condition is: $f(w_R) - f(w_L) = S(w_R - w_L)$, with $f(w) = \frac{2}{3}w^{3/2}$
   $$\frac{2}{3}w_R^{3/2} - \frac{2}{3}w_L^{3/2} = S(w_R - w_L)$$
   $$S = \frac{2}{3}\frac{w_R^{3/2} - w_L^{3/2}}{w_R - w_L} > 0$$

   (d) Entropy condition: $\lambda(w_L) > S > \lambda(w_R)$ for shock formation. $\lambda(w_L) < \lambda(w_R)$ for rarefaction. The head and tail of the rarefaction fan are given by $\lambda(w_L)$ and $\lambda(w_R)$. Within the fan, $\lambda(w)$ take any values. $\lambda(w) > 0$, so only right-moving waves.



   (e) Primitive variable form:
   $$\partial_t(u^2) + \partial_x\left(\frac{2}{3}u^3\right) = 0$$
   $$2u\partial_t u + 2u^2\partial_x u = 0$$
   $$\partial_t u + u\partial_x u = 0$$

   (f) They can be written in the same primitive variable (non-conservative) form.

   (g) Same Riemann problem solution of right-moving shock and rarefaction wave. The shock speeds are different, for burger's equation, $\hat{S} = \frac{1}{2}(\hat{u}_L + \hat{u}_R)$. Rarefaction head and tail remains unchanged in u and $\hat{u}$ [Burger's: $\hat{\lambda}(\hat{u}) = \hat{u}$, equation (1): $\lambda(w) = \sqrt{w}$ but $\lambda(u) = u$]

(h) If you write things in primitive form, you lose sight of the flux function, and you could be solving either equation (1) or Burger's equation (or a whole family of equations). If a discontinuity is obtained, then identifying which equation caused it requires knowledge of the jump conditions (RH conditions uses flux information).

Smooth solutions (rarefactions) are unchanged in either form. The shock speed becomes unclear in non-conservative form, whereas the conservation law form makes it very clear the corresponding shock speed.

7.2 Q2

2. (a)

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{\Delta x}\left(\beta_1(u_i^n - u_{i-1}^n) + \beta_2(u_{i+1}^n - u_i^n)\right)$$

$$u_i^{n+1} = u_i^n - C(u_i^n(\beta_1 - \beta_2) - \beta_1 u_{i-1}^n + \beta_2 u_{i+1}^n)$$

$$u_i^{n+1} = (1 - C^2)u_i^n + \frac{1}{2}C(1 + C)u_{i-1}^n - \frac{1}{2}C(1 - C)u_{i+1}^n$$

(b) Lax-Wendroff method.

(c) CFL number. This number is directly related to the stability of the numerical method. It informs of a stable time-step given a certain Δx. For Lax-Wendroff, $|C| < 1$ for the error to not increase with every time-step.

(d)

$$\begin{aligned}
u_i^{n+1} &= u_i^n - C^2 u_i^n + \frac{1}{2}C(1 + C)u_{i-1}^n - \frac{1}{2}C(1 - C)u_{i+1}^n \\
&= u_i^n - \frac{1}{2}C^2 u_i^n - \frac{1}{2}C^2 u_i^n + \frac{1}{2}Cu_{i-1}^n + \frac{1}{2}C^2 u_{i-1}^n - \frac{1}{2}Cu_{i+1}^n + \frac{1}{2}C^2 u_{i+1}^n \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}aCu_i^n + \frac{1}{2}aCu_i^n + \frac{1}{2}au_i^n - \frac{1}{2}au_i^n - \frac{1}{2}au_{i-1}^n - \frac{1}{2}aCu_{i-1}^n + \frac{1}{2}au_{i+1}^n - \frac{1}{2}aCu_{i+1}^n\right) \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}\left[au_{i+1}^n + au_i^n\right] - \frac{1}{2}\left[au_{i-1}^n + au_i^n\right] - \frac{1}{2}aC\left[u_{i+1}^n - u_i^n\right] + \frac{1}{2}aC(u_i^n - u_{i-1}^n)\right) \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left(\left[\frac{1}{2}(au_{i+1}^n + au_i^n) - \frac{a^2\Delta t}{2\Delta x}(u_{i+1}^n - u_i^n)\right] - \left[\frac{1}{2}(au_i^n + au_{i-1}^n) - \frac{a^2\Delta t}{2\Delta x}(u_i^n - u_{i-1}^n)\right]\right)
\end{aligned}$$

$$f_{i+1/2} = \frac{1}{2}(au_{i+1}^n + au_i^n) - \frac{a^2\Delta t}{2\Delta x}(u_{i+1}^n - u_i^n)$$

(e) Since u_i^{n+1} and N_i^{n+1} satisfy the numerical method, then ε^{n+1} will satisfy the numerical method too.

(f)

$$E_m(t + \Delta t) = E_m(t) - C^2 E_m(t) + \frac{1}{2}C(1 + C)E_m(t)e^{-ik_m\Delta x} - \frac{1}{2}C(1 - C)E_m(t)e^{ik_m\Delta x}$$

$$\begin{aligned}
\Delta E &= \frac{E_m(t + \Delta t)}{E_m(t)} \\
&= 1 - C^2 + \frac{1}{2}C(1 + C)e^{-ik_m\Delta x} - \frac{1}{2}C(1 - C)e^{ik_m\Delta x} \\
&= (1 - C^2) - \frac{1}{2}C\left(e^{ik_m\Delta x} - e^{-ik_m\Delta x}\right) + \frac{1}{2}C^2\left(e^{ik_m\Delta x} + e^{-ik_m\Delta x}\right) \\
&= (1 - C^2) + C^2\cos(k_m\Delta x) - iC\sin(k_m\Delta x)
\end{aligned}$$

(g)

$$|\Delta E|^2 = [C^2(\cos(k_m\Delta x) - 1) + 1]^2 + C^2\sin^2(k_m\Delta x) \leq 1$$
$$1 + C^4(\cos(k_m\Delta x) - 1)^2 + 2C^2(\cos(k_m\Delta x) - 1) + C^2\sin^2(k_m\Delta x) \leq 1$$
$$C^2(\cos(k_m\Delta x) - 1)^2 + 2(\cos(k_m\Delta x) - 1) + \sin^2(k_m\Delta x) \leq 0$$
$$C^2(\cos(k_m\Delta x) - 1)^2 + 2(\cos(k_m\Delta x) - 1) + 1 - \cos^2(k_m\Delta x) \leq 0$$
$$C^2(\cos(k_m\Delta x) - 1)^2 - (\cos(k_m\Delta x) - 1)^2 \leq 0$$
$$(C^2 - 1)(\cos(k_m\Delta x) - 1)^2 \leq 0$$
$$C^2 - 1 \leq 0$$
$$C^2 \leq 1$$
$$|C| \leq 1$$

Von Neumann Stability Analysis

$$u_i^{n+1} = \underbrace{N_i^{n+1}}_{\substack{\text{numerical}\\\text{solution}}} + \underbrace{\epsilon_i^{n+1}}_{\substack{\text{error}\\\text{term}}}$$

Since the numerical solution satisfies the update formula, the error term must satisfy too.

We assume discrete Fourier series exists for ϵ:

$$\epsilon(x,t) = \sum_{m=-M}^{M} E_m(t)e^{ik_m x} \quad \text{where } k_m = \frac{2\pi m}{L}$$

Note, in a discrete domain with 2M + 1 points, there are a finite number of frequencies possible, $m \in [-M, M]$. Substitute the Fourier expansion for ε into the update formula:

$$\epsilon_{i,m}^{n+1} = E_m(t+\Delta t)e^{ik_m x} \quad , \quad \epsilon_{i,m}^{n} = E_m(t)e^{ik_m x} \quad , \quad \epsilon_{i+1,m}^{n} = E_m(t)e^{ik_m(x+\Delta x)}$$

Then, we define the change in error over a time step:

$$\Delta E_m = \frac{E_m(t+\Delta t)}{E_m(t)}$$

where it may be complex. For stability,

$$|\Delta E|^2 \leq 1$$

This will give us the CFL condition:

$$C = \frac{|a_{\max}|}{\Delta x/\Delta t}$$

We can also define the total error:

$$E_m^{tot} = \sum_{n=1}^{T} \frac{E_m(t^n)}{E_m(t^{n-1})}$$

We can minimise the total error by ensuring T is as small as possible meaning we want to take the fewest time steps possible. We want C to be as large as possible.

7.3 Q3

3. (a) Conservative form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho v\\ \rho v^2+p\\ [E+p]v\end{pmatrix}=0$$

Expand the conservative form and deduce the matrix B to get the primitive form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ v\\ p\end{pmatrix}+\underbrace{\begin{pmatrix}v & \rho & 0\\ 0 & v & 1/\rho\\ 0 & \rho c_s^2 & v\end{pmatrix}}_{B}\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ v\\ p\end{pmatrix}=0$$

(b) A is the Jacobian.

$$A_{ij}=\frac{\partial f_i}{\partial u_j}$$

(c)

$$p=(\gamma-1)\rho\varepsilon \quad \text{and} \quad E=\rho\varepsilon+\frac{1}{2}\rho v^2$$

$$p=(\gamma-1)\left(E-\frac{1}{2}\rho v^2\right)=(\gamma-1)\left(E-\frac{1}{2}\frac{(\rho v)^2}{\rho}\right)$$

$$\begin{cases} f_1=\rho v\\ f_2=\frac{(\rho v)^2}{\rho}+(\gamma-1)\left(E-\frac{1}{2}\frac{(\rho v)^2}{\rho}\right)\\ f_3=E\frac{\rho v}{\rho}+\frac{\rho v}{\rho}(\gamma-1)\left(E-\frac{1}{2}\frac{(\rho v)^2}{\rho}\right)=\frac{E\rho v}{\rho}+(\gamma-1)\left(E\frac{\rho v}{\rho}-\frac{1}{2}\frac{(\rho v)^3}{\rho^2}\right)\end{cases}$$

$$\boldsymbol{u}=\begin{pmatrix}\rho & \rho v & E\end{pmatrix}^T$$

$$\begin{aligned}A_{ij}&=\begin{pmatrix}\partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \partial f_1/\partial u_3\\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \partial f_2/\partial u_3\\ \partial f_3/\partial u_1 & \partial f_3/\partial u_2 & \partial f_3/\partial u_3\end{pmatrix}\\
&=\begin{pmatrix}0 & 1 & 0\\ -\frac{(\rho v)^2}{\rho^2}+\frac{1}{2}(\gamma-1)\frac{(\rho v)^2}{\rho^2} & 2v-(\gamma-1)v & \gamma-1\\ -\frac{E\rho v}{\rho^2}+(\gamma-1)(-\frac{E\rho v}{\rho^2}+\frac{(\rho v)^3}{\rho^3}) & \frac{E}{\rho}+(\gamma-1)(\frac{E}{\rho}-\frac{3}{2}\frac{(\rho v)^2}{\rho^2}) & v+(\gamma-1)v\end{pmatrix}\\
&=\begin{pmatrix}0 & 1 & 0\\ -v^2+\frac{1}{2}(\gamma-1)v^2 & 3v-\gamma v & \gamma-1\\ -\frac{Ev}{\rho}+(\gamma-1)(-\frac{Ev}{\rho}+v^3) & \frac{E}{\rho}+(\gamma-1)(\frac{E}{\rho}-\frac{3}{2}v^2) & \gamma v\end{pmatrix}\\
&=\begin{pmatrix}0 & 1 & 0\\ \frac{1}{2}(\gamma-3)v^2 & (3-\gamma)v & \gamma-1\\ -\gamma\frac{Ev}{\rho}+(\gamma-1)v^3 & \gamma\frac{E}{\rho}-\frac{3}{2}(\gamma-1)v^2 & \gamma v\end{pmatrix}\end{aligned}$$

(d) A and B are similar matrices. They are related by $A = C^{-1}BC$ or $B = CAC^{-1}$.

$$\frac{\partial \mathbf{w}}{\partial t} + B\frac{\partial \mathbf{w}}{\partial t} = 0$$

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{w}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial x} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + C^{-1}BC\frac{\partial \mathbf{u}}{\partial x} = 0 \quad \text{, where} \quad C_{ij} = \frac{\partial w_i}{\partial u_j}$$

(e)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + p)}{\partial x} = 0$$

$$\frac{\partial(E)}{\partial t} + \frac{\partial(E+p)v}{\partial x} = 0$$

with $E = \rho\varepsilon + \frac{1}{2}\rho v^2$.

$$\frac{\partial(\rho\varepsilon)}{\partial t} + \frac{1}{2}\frac{\partial(\rho v^2)}{\partial t} + \frac{\partial(\rho\varepsilon v)}{\partial x} + \frac{1}{2}\frac{\partial(\rho v^3)}{\partial x} + \frac{\partial(vp)}{\partial x} = 0$$

$$\rho\frac{\partial \varepsilon}{\partial t} + \cancel{\varepsilon\frac{\partial \rho}{\partial t}} + \frac{1}{2}\frac{\partial(\rho v^2)}{\partial t} + \cancel{\varepsilon\frac{\partial(\rho v)}{\partial x}} + \rho v\frac{\partial \epsilon}{\partial x} + \frac{1}{2}\frac{\partial(\rho v^3)}{\partial x} + \frac{\partial(vp)}{\partial x} = 0$$

$$\rho\frac{\partial \varepsilon}{\partial t} + \rho v\frac{\partial \epsilon}{\partial x} + \frac{1}{2}\left(\cancel{v\frac{\partial(\rho v)}{\partial t}} + \rho v\frac{\partial v}{\partial t} + \cancel{v\frac{\partial(\rho v^2)}{\partial x}} + \rho v^2\frac{\partial v}{\partial x} + \cancel{v\frac{\partial p}{\partial x}} - v\frac{\partial p}{\partial x}\right) + \frac{\partial(vp)}{\partial x} = 0$$

$$\rho\frac{\partial \varepsilon}{\partial t} + \rho v\frac{\partial \epsilon}{\partial x} + \frac{1}{2}\left(\rho v\frac{\partial v}{\partial t} + \rho v^2\frac{\partial v}{\partial x} - v\frac{\partial p}{\partial x} + v^2\frac{\partial \rho}{\partial t} + v^2\frac{\partial(\rho v)}{\partial x}\right) + \frac{\partial(vp)}{\partial x} = 0$$

$$\rho\frac{\partial \varepsilon}{\partial t} + \rho v\frac{\partial \epsilon}{\partial x} + \frac{1}{2}\left(v\cancel{\left[\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + \rho v\frac{\partial v}{\partial x} + v\frac{\partial(\rho v)}{\partial x} + \frac{\partial p}{\partial x}\right]} - 2v\frac{\partial p}{\partial x}\right) + \frac{\partial(vp)}{\partial x} = 0$$

$$\rho\frac{\partial \varepsilon}{\partial t} + \rho v\frac{\partial \epsilon}{\partial x} \cancel{- v\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial x}} + p\frac{\partial v}{\partial x} = 0$$

$$\rho\frac{\partial \varepsilon}{\partial t} + \rho v\frac{\partial \epsilon}{\partial x} + p\frac{\partial v}{\partial x} = 0$$

(f) Density:

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial v}{\partial x} + v\frac{\partial \rho}{\partial x} = 0$$

Momentum:

$$\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + 2\rho v\frac{\partial v}{\partial x} + v^2\frac{\partial \rho}{\partial x} + \frac{\partial p}{\partial x} = 0$$

$$\Rightarrow \rho\frac{\partial v}{\partial t} - v\rho\frac{\partial v}{\partial x} - \cancel{v^2\frac{\partial \rho}{\partial x}} + 2\rho v\frac{\partial v}{\partial x} + \cancel{v^2\frac{\partial \rho}{\partial x}} + \frac{\partial p}{\partial x} = 0$$

$$\Rightarrow \rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial p}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x}\right) = 0$$

Energy:

$$\frac{\partial \varepsilon}{\partial t} + v\frac{\partial \epsilon}{\partial x} + \frac{p}{\rho}\frac{\partial v}{\partial x} = 0$$

So, in primitive form:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ v \\ \varepsilon\end{pmatrix} + \begin{pmatrix} v & \rho & 0 \\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & v & \frac{1}{\rho}\frac{\partial p}{\partial \varepsilon} \\ 0 & \frac{p}{\rho} & v \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho \\ v \\ \varepsilon\end{pmatrix}$$

(g)

$$\begin{vmatrix} v-\lambda & \rho & 0 \\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & v-\lambda & \frac{1}{\rho}\frac{\partial p}{\partial \varepsilon} \\ 0 & \frac{p}{\rho} & v-\lambda \end{vmatrix} = 0$$

$$(v-\lambda)\left((v-\lambda)^2 - \frac{p}{\rho^2}\frac{\partial p}{\partial \varepsilon}\right) - \rho\left(\frac{v-\lambda}{\rho}\frac{\partial p}{\partial \rho}\right) = 0$$

$$\lambda_0 = v$$

$$(v-\lambda)^2 - \frac{p}{\rho^2}\frac{\partial p}{\partial \varepsilon} - \frac{\partial p}{\partial \rho} = 0$$

$$(v-\lambda)^2 = c_s^2$$

$$v-\lambda = \pm c_s$$

$$\lambda = v \pm c_s$$

Eigenvalues: $\lambda = v - c_s, v, v + c_s$.

8 2023-24 (EXAM)

8.1 Q1

1. Same question as 2024-25 (Mock) Q1. Link

8.2 Q2

2. (a)

$$E = \rho\varepsilon + \frac{1}{2}\rho v^2$$

(b) Standard sound speed derivation:

$$c_s^2 = \frac{\partial p}{\partial \rho}\Big|_s$$

Using $p = p(\varepsilon(\rho, s), \rho)$,

$$dp = \frac{\partial p}{\partial \varepsilon}\Big|_\rho d\varepsilon + \frac{\partial p}{\partial \rho}\Big|_\varepsilon d\rho = \frac{\partial p}{\partial \varepsilon}\Big|_\rho \left(\frac{\partial \varepsilon}{\partial \rho}\Big|_s d\rho + \frac{\partial \varepsilon}{\partial s}\Big|_\rho ds\right) + \frac{\partial p}{\partial \rho}\Big|_\varepsilon d\rho$$

Now, let's keep s constant,

$$dp = \frac{\partial p}{\partial \varepsilon}\Big|_\rho \left(\frac{\partial \varepsilon}{\partial \rho}\Big|_s d\rho + \cancel{\frac{\partial \varepsilon}{\partial s}\Big|_\rho ds}\right) + \frac{\partial p}{\partial \rho}\Big|_\varepsilon d\rho$$

Then, one can easily see,

$$\frac{\partial p}{\partial \rho}\Big|_s = \frac{\partial p}{\partial \varepsilon}\Big|_\rho \frac{\partial \varepsilon}{\partial \rho}\Big|_s + \frac{\partial p}{\partial \rho}\Big|_\varepsilon$$

We use $Tds = d\varepsilon + pd\nu$ to find $\frac{\partial \varepsilon}{\partial \rho}\big|_s$,

$$d\varepsilon + pd\nu = 0$$

$$p = -\frac{\partial \varepsilon}{\partial \nu}\Big|_s = \rho^2 \frac{\partial \varepsilon}{\partial \rho}\Big|_s$$

$$\Rightarrow \frac{\partial \varepsilon}{\partial \rho}\Big|_s = \frac{p}{\rho^2}$$

Therefore,

$$c_s^2 = \frac{p}{\rho^2}\frac{\partial p}{\partial \varepsilon}\Big|_\rho + \frac{\partial p}{\partial \rho}\Big|_\varepsilon$$

Otherwise, let's say if we have this instead $\varepsilon = \varepsilon(p, \rho) = \varepsilon(p(\rho, s), \rho)$. Same thing,

$$d\varepsilon = \frac{\partial \varepsilon}{\partial p}\bigg|_\rho dp + \frac{\partial \varepsilon}{\partial \rho}\bigg|_p d\rho = \frac{\partial \varepsilon}{\partial p}\bigg|_\rho \left(\frac{\partial p}{\partial \rho}\bigg|_s d\rho + \frac{\partial p}{\partial s}\bigg|_\rho ds\right) \frac{\partial \varepsilon}{\partial \rho}\bigg|_p d\rho$$

Let's keep s constant,

$$d\varepsilon = \frac{\partial \varepsilon}{\partial p}\bigg|_\rho \left(\frac{\partial p}{\partial \rho}\bigg|_s d\rho + \cancel{\frac{\partial p}{\partial s}\bigg|_\rho ds}\right) \frac{\partial \varepsilon}{\partial \rho}\bigg|_p d\rho$$

Clearly,

$$\frac{\partial \varepsilon}{\partial \rho}\bigg|_s = \frac{\partial \varepsilon}{\partial p}\bigg|_\rho \frac{\partial p}{\partial \rho}\bigg|_s + \frac{\partial \varepsilon}{\partial \rho}\bigg|_p$$

We use $Tds = d\varepsilon + pd\nu$ to find $\frac{\partial \varepsilon}{\partial \rho}\big|_s$,

$$\frac{\partial \varepsilon}{\partial \rho}\bigg|_s = \frac{p}{\rho^2}$$

Therefore,

$$\frac{p}{\rho^2} = \frac{\partial \varepsilon}{\partial p}\bigg|_\rho c_s^2 + \frac{\partial \varepsilon}{\partial \rho}\bigg|_p$$

$$c_s^2 = \frac{p}{\rho^2 \frac{\partial \varepsilon}{\partial p}\big|_\rho} - \frac{\frac{\partial \varepsilon}{\partial \rho}\big|_p}{\frac{\partial \varepsilon}{\partial p}\big|_\rho}$$

(c) Perturbation of conservation of mass equation:

$$\frac{\partial}{\partial t}(\rho_0 + \bar{\rho}) + \frac{\partial}{\partial x}[(\rho_0 + \bar{\rho})\bar{v}] = 0$$

$$\cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho_0 \bar{v}}{\partial x} + \cancel{\frac{\partial \bar{\rho}\bar{v}}{\partial x}} = 0$$

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho_0 \bar{v}}{\partial x} = 0$$

(d) Perturbation of conservation of energy:

$$\begin{aligned} E &= (\rho_0 + \bar{\rho})(\varepsilon_0 + \bar{\varepsilon}) + \cancel{\frac{1}{2}(\rho_0 + \bar{\rho})\bar{v}^2} \\ &= \rho_0\varepsilon_0 + \bar{\rho}\varepsilon_0 + \rho_0\bar{\varepsilon} \end{aligned}$$

Then,

$$\frac{\partial}{\partial t}(\cancel{\rho_0\varepsilon_0} + \bar{\rho}\varepsilon_0 + \rho_0\bar{\varepsilon}) + \frac{\partial}{\partial x}[(\rho_0\varepsilon_0 + \cancel{\bar{\rho}\varepsilon_0} + \cancel{\rho_0\bar{\varepsilon}} + p_0 + \cancel{\bar{p}})\bar{v}] = 0$$

$$\frac{\partial}{\partial t}(\bar{\rho}\varepsilon_0 + \rho_0\bar{\varepsilon}) + \frac{\partial}{\partial x}[(\rho_0\varepsilon_0 + p_0 +)\bar{v}] = 0$$

$$\cancel{\varepsilon_0 \frac{\partial \bar{\rho}}{\partial t} + \varepsilon_0 \frac{\partial \rho_0 \bar{v}}{\partial x}} + \rho_0 \frac{\partial \bar{\varepsilon}}{\partial t} + \frac{\partial p_0 \bar{v}}{\partial x} = 0$$

$$\frac{\partial \bar{\varepsilon}}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p_0}{\rho_0}\bar{v}\right) = 0$$

(e) Since ε is a function of ρ, we compare the perturbed conservation of mass and energy equation to see that they are related to each other by a constant factor $\frac{p_0}{\rho_0^2}$,

$$\frac{p_0}{\rho_0^2}\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p_0}{\rho_0^2}\rho_0 \bar{v}\right) = 0$$

$$\frac{p_0}{\rho_0^2}\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p_0}{\rho_0}\bar{v}\right) = 0$$

Hence,

$$d\bar{\varepsilon} = \frac{p_0}{\rho_0^2} d\bar{\rho}$$

Integrating gives:

$$\bar{\varepsilon} = \frac{p_0}{\rho_0^2}\bar{\rho} + A$$

(f)

$$\begin{aligned} c_s^2 &= \frac{p}{\rho^2}\frac{\partial p}{\partial \varepsilon}\bigg|_{\rho} + \frac{\partial p}{\partial \rho}\bigg|_{\varepsilon} \\ &= \frac{p_0 + \bar{p}}{(\rho_0 + \bar{\rho})^2}\frac{\partial (p_0 + \bar{p})}{\partial(\varepsilon_0 + \bar{\varepsilon})}\bigg|_{\rho_0 + \bar{\rho}} + \frac{\partial (p_0 + \bar{p})}{\partial(\rho_0 + \bar{\rho})}\bigg|_{\varepsilon_0 + \bar{\varepsilon}} \end{aligned}$$

$$\begin{aligned} \frac{p_0 + \bar{p}}{(\rho_0 + \bar{\rho})^2} &= \frac{p_0 + \bar{p}}{\rho_0^2}\left(1 + \frac{\bar{\rho}}{\rho_0}\right)^{-2} \\ &= \frac{p_0 + \bar{p}}{\rho_0^2}\left(1 - 2\frac{\bar{\rho}}{\rho_0}\right) \\ &= \frac{p_0}{\rho_0^2} + \frac{\bar{p}}{\rho_0^2} - 2\frac{p_0\bar{\rho}}{\rho_0^3} \end{aligned}$$

The partial derivatives reduce to derivative of perturbed quantities hence, we keep only the zeroth order part of the pre-factor constant. This gives us the required expression for c_s:

$$c_s = \sqrt{\frac{p_0}{\rho_0^2}\frac{\partial \bar{p}}{\partial \bar{\varepsilon}}\bigg|_{\bar{\rho}} + \frac{\partial \bar{p}}{\partial \bar{\rho}}\bigg|_{\bar{\varepsilon}}}$$

(g) Perturbation of conservation of momentum:

$$\frac{\partial}{\partial t}[(\rho_0 + \cancel{\bar{\rho}})\bar{v}] + \frac{\partial}{\partial x}\left[\cancel{(\rho_0 + \bar{\rho})\bar{v}^2} + \cancel{p_0} + \bar{p}\right]$$

$$\frac{\partial \rho_0 \bar{v}}{\partial t} + \frac{\partial \bar{p}}{\partial x} = 0$$

We hope to show that $\bar{p} = c_s^2\bar{\rho}$. Since $\bar{\varepsilon}(p, \rho) = \frac{p_0}{\rho_0^2}\bar{\rho} + A$,

$$\Rightarrow \frac{\partial \bar{p}}{\partial \bar{\varepsilon}}\bigg|_{\bar{\rho}} = 0$$

Then,

$$c_s^2 = \frac{\partial \bar{p}}{\partial \bar{\rho}}\bigg|_{\bar{\varepsilon}}$$

Integrating gives,

$$c_s^2 \bar{\rho} = \bar{p} + C$$

Substituting back,

$$\frac{\partial \rho_0 \bar{v}}{\partial t} + \frac{\partial}{\partial x}(c_s^2 \bar{\rho} - C) = 0$$

$$\frac{\partial \bar{v}}{\partial t} + \frac{\partial}{\partial x}\left(\frac{c_s^2}{\rho_0}\bar{\rho}\right) = 0$$

(h) Differentiating w.r.t. space,

$$\frac{\partial}{\partial t}\frac{\partial \bar{v}}{\partial x} + \frac{c_s^2}{\rho_0}\frac{\partial^2 \bar{\rho}}{\partial x^2} = 0$$

$$\frac{\partial}{\partial t}\left(-\frac{1}{\rho_0}\frac{\partial \bar{\rho}}{\partial t}\right) + \frac{c_s^2}{\rho_0}\frac{\partial^2 \bar{\rho}}{\partial x^2} = 0$$

$$\frac{\partial^2 \bar{\rho}}{\partial t^2} = \frac{\partial^2}{\partial x^2}(c_s^2 \bar{\rho})$$

(i) Wave equation with speed c_s.

8.3 Q3

3. (a) Consider,

$$\begin{vmatrix} v-\lambda & \rho & 0 \\ \frac{c_s^2}{\rho} & v-\lambda & \frac{1}{\rho}\frac{\partial p}{\partial s} \\ 0 & 0 & v-\lambda \end{vmatrix} = 0$$

$$(v-\lambda)^3 - \rho(v-\lambda)\frac{c_s^2}{\rho} = 0$$

$$(v-\lambda)\left((v-\lambda)^2 - c_s^2\right) = 0$$

$$\lambda = v\,,\ v+c_s\,,\ v-c_s$$

(b) $A\boldsymbol{r} = \lambda\boldsymbol{r}$.

$$\begin{pmatrix} v & \rho & 0 \\ \frac{c_s^2}{\rho} & v & \frac{1}{\rho}\frac{\partial p}{\partial s} \\ 0 & 0 & v \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{cases} av + \rho b = \lambda a \\ a\frac{c_s^2}{\rho} + bv + c\frac{1}{\rho}\frac{\partial p}{\partial s} = \lambda b \\ vc = \lambda c \end{cases}$$

For $\lambda = v$,

$$\begin{cases} \cancel{av} + \rho b = \cancel{av} \\ a\frac{c_s^2}{\rho} + \cancel{bv} + c\frac{1}{\rho}\frac{\partial p}{\partial s} = \cancel{bv} \\ vc = vc \end{cases}$$

Solution is $b = 0, c = c_s^2, a = -\frac{\partial p}{\partial s}$. The corresponding right eigenvector for $\lambda = v$ is

$$\boldsymbol{r}^2 = \begin{pmatrix} -\frac{\partial p}{\partial s} \\ 0 \\ c_s^2 \end{pmatrix}$$

For $\lambda = v + c_s$,

$$\begin{cases} \cancel{av} + \rho b = a(\cancel{v} + c_s) \\ a\frac{c_s^2}{\rho} + \cancel{bv} + c\frac{1}{\rho}\frac{\partial p}{\partial s} = b(\cancel{v} + c_s) \\ \cancel{vc} = c(\cancel{v} + c_s) \end{cases}$$

Solution is $c = 0, a = 1, \frac{c_s}{\rho}$. The corresponding right eigenvector for $\lambda = v + c_s$ is

$$\boldsymbol{r}^3 = \begin{pmatrix} 1 \\ \frac{c_s}{\rho} \\ 0 \end{pmatrix}$$

For $\lambda = v - c_s$,

$$\begin{cases} \cancel{av} + \rho b = a(\cancel{v} - c_s) \\ a\frac{c_s^2}{\rho} + \cancel{bv} + c\frac{1}{\rho}\frac{\partial p}{\partial s} = b(\cancel{v} - c_s) \\ \cancel{vc} = c(\cancel{v} - c_s) \end{cases}$$

Solution is $c = 0, a = 1, -\frac{c_s}{\rho}$. The corresponding right eigenvector for $\lambda = v - c_s$ is

$$\boldsymbol{r}^1 = \begin{pmatrix} 1 \\ -\frac{c_s}{\rho} \\ 0 \end{pmatrix}$$

(c)

$$\frac{d\rho}{1} = \frac{dv}{-c_s/\rho} = \frac{ds}{0}$$

Taking any pairs of equality with the ds term gives:

$$\Rightarrow ds = 0$$

(d)

$$s = c_V \ln p - c_p \ln \rho + \text{const}$$
$$ds = \frac{c_V}{p} dp - \frac{c_p}{\rho} d\rho$$

For $ds = 0$ and we know that $\frac{c_p}{c_V} = \gamma$,

$$\frac{\gamma c_V}{\rho} d\rho = \frac{c_V}{p} dp$$
$$\frac{\gamma d\rho}{\rho} = \frac{dp}{p}$$

Integrating

$$\gamma \ln \rho = \ln p + \text{const}$$
$$\Rightarrow p = e^{\text{const}} \rho^\gamma = A\rho^\gamma$$

(e) Consider,

$$d\rho + \frac{\rho}{c_s} dv = 0$$
$$dv + \frac{c_s}{\rho} d\rho = 0$$

Integrating and shifting all constants to the RHS

$$v + \int \frac{c_s}{\rho} d\rho = \text{const}$$

Noting that $c_s^2 = \frac{\partial p}{\partial \rho}\Big|_s$

$$c_s^2 = A\gamma\rho^{\gamma-1}$$
$$c_s = \sqrt{A\gamma}\rho^{(\gamma-1)/2} \tag{2}$$

Then,

$$\int \frac{c_s}{\rho} d\rho = \sqrt{A\gamma} \int \rho^{\gamma/2 - 3/2} d\rho$$
$$= \sqrt{A\gamma} \frac{\rho^{\gamma/2 - 1/2}}{\gamma/2 - 1/2}$$
$$= \frac{2c_s}{\gamma - 1}$$

Hence,

$$v + \frac{2c_s}{\gamma - 1} = \text{const}$$

(f) Across a rarefaction,

$$\frac{p}{\rho^\gamma} = \text{const}$$
$$\frac{p_L}{\rho_L^\gamma} = \frac{p^*}{\rho_L^{*\gamma}}$$
$$\rho_L^{*\gamma} = \rho_L^\gamma \frac{p^*}{p_L}$$
$$\rho_L^* = \rho_L \left(\frac{p^*}{p_L}\right)^{1/\gamma}$$

(g)

$$v_L + \frac{2c_{s,L}}{\gamma - 1} = v^* + \frac{2c_s^*}{\gamma - 1}$$

From equation (2),

$$\frac{c_{s,L}}{c_s^*} = \left(\frac{\rho_L}{\rho_L^*}\right)^{(\gamma-1)/2} = \left(\frac{p_L}{p^*}\right)^{(\gamma-1)/2\gamma}$$
$$\Rightarrow c_s^* = c_{s,L}\left(\frac{p^*}{p_L}\right)^{(\gamma-1)/2\gamma}$$

Then,

$$v^* = v_L + \frac{2c_{s,L}}{\gamma - 1} - \frac{2c_{s,L}}{\gamma - 1}\left(\frac{p^*}{p_L}\right)^{(\gamma-1)/2\gamma}$$
$$v^* = v_L - \frac{2c_{s,L}}{\gamma - 1}\left[\left(\frac{p^*}{p_L}\right)^{(\gamma-1)/2\gamma} - 1\right]$$

(h) Equating and making p^* the subject,

$$v_L - \frac{2c_{s,L}}{\gamma - 1}\left[\left(\frac{p^*}{p_L}\right)^{(\gamma-1)/2\gamma} - 1\right] = v_R + \frac{2c_{s,R}}{\gamma - 1}\left[\left(\frac{p^*}{p_R}\right)^{(\gamma-1)/2\gamma} - 1\right]$$
$$v_L - v_R + \frac{2c_{s,L}}{\gamma - 1} + \frac{2c_{s,R}}{\gamma - 1} = \frac{2c_{s,R}}{\gamma - 1}\left(\frac{p^*}{p_R}\right)^{(\gamma-1)/2\gamma} + \frac{2c_{s,L}}{\gamma - 1}\left(\frac{p^*}{p_L}\right)^{(\gamma-1)/2\gamma}$$
$$\frac{(\gamma-1)(v_L - v_R)}{2} + c_{s,L} + c_{s,R} = \left[\frac{c_{s,R}}{p_R^{(\gamma-1)/2\gamma}} + \frac{c_{s,L}}{p_L^{(\gamma-1)/2\gamma}}\right] p^{*(\gamma-1)/2\gamma}$$
$$p^{*(\gamma-1)/2\gamma} = \frac{\frac{(\gamma-1)(v_L - v_R)}{2} + c_{s,L} + c_{s,R}}{\left[\frac{c_{s,R}}{p_R^{(\gamma-1)/2\gamma}} + \frac{c_{s,L}}{p_L^{(\gamma-1)/2\gamma}}\right]}$$
$$p^* = \left[\frac{\frac{1}{2}(\gamma-1)(v_L - v_R) + (c_{s,L} + c_{s,R})}{\left[\frac{c_{s,R}}{p_R^{(\gamma-1)/2\gamma}} + \frac{c_{s,L}}{p_L^{(\gamma-1)/2\gamma}}\right]}\right]^{2\gamma/(\gamma-1)}$$

9 2024-25 (MOCK)

9.1 Q1

1. (a) k_L and k_R are integers that determine left and right bounds of the numerical stencil.

   (b)
$$C = a\frac{\Delta t}{\Delta x} \quad \text{or}$$

   Monotone if:
$$\frac{\partial H}{\partial u_j^n} \geq 0 \quad , \forall\, j$$

   For LF scheme: $u_i^{n+1} = \frac{1}{2}(1+C)u_{i-1}^n + \frac{1}{2}(1-C)u_{i+1}^n$,
$$\frac{\partial u_i^{n+1}}{\partial u_{i-1}^n} = \frac{1}{2}(1+C) \geq 0 \quad \text{and} \quad \frac{\partial u_i^{n+1}}{\partial u_{i+1}^n} = \frac{1}{2}(1-C) \geq 0$$
$$\Rightarrow -1 \leq C \leq 1$$

   (c) Burger's equation (conservation form):
$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

   Applying backward differencing:
$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_i^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

   Consider:
$$\frac{\partial H}{\partial u_i^n} = 1 - \frac{\Delta t}{\Delta x}u_i^n \geq 0 \quad \text{and} \quad \frac{\partial H}{\partial u_{i-1}^n} = \frac{\Delta t}{\Delta x}u_{i-1}^n \geq 0$$

   Monotone when:
$$u_{i-1}^n \geq 0 \quad \text{and} \quad u_i^n \leq \frac{\Delta x}{\Delta t}$$

   So,
$$\boxed{0 \leq u_i^n \leq \frac{\Delta x}{\Delta t}}$$

   (d) Burger's equation (primitive form):
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

   Applying backward differencing:
$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left((u_i^n)^2 - u_i^n \cdot u_{i-1}^n\right)$$

   Consider:
$$\frac{\partial H}{\partial u_i^n} = 1 - \frac{\Delta t}{\Delta x}(2u_i^n - u_{i-1}^n) \geq 0 \quad \text{and} \quad \frac{\partial H}{\partial u_{i-1}^n} = \frac{\Delta t}{\Delta x}u_i^n \geq 0$$

$$2u_i^n - u_{i-1}^n \le \frac{\Delta x}{\Delta t} \quad \text{and} \quad u_i^n \ge 0$$

Monotone when:

$$u_{i-1}^n \ge 2u_i^n - \frac{\Delta x}{\Delta t} \quad \text{and} \quad u_i^n \ge 0$$

So,

$$\boxed{0 \le u_i^n \le \frac{\Delta x}{\Delta t}}$$

(e) Monotone when:

$$\begin{cases} u_{i+1}^n \le 0 \quad \text{and} \quad u_i^n \ge -\frac{\Delta x}{\Delta t} \quad \text{(conservative)} \\ u_i^n \le 0 \quad \text{and} \quad u_{i+1}^n \le 2u_i^n + \frac{\Delta x}{\Delta t} \quad \text{(primitive)} \end{cases}$$

So,

$$\boxed{-\frac{\Delta x}{\Delta t} \le u_i^n \le 0}$$

(f) Given numerical flux using Burger's equation:

$$f_{i+1/2}^n = \frac{1}{2}\frac{\Delta x}{\Delta t}(u_i^n - u_{i+1}^n) + \frac{1}{2}\left(\frac{1}{2}(u_{i+1}^n)^2 + \frac{1}{2}(u_i^n)^2\right)$$

Consider:

$$\begin{aligned} \frac{\partial f_{i+1/2}^n}{\partial u_i^n} =& \frac{1}{2}\frac{\Delta x}{\Delta t} + \frac{1}{2}(u_i^n) \ge 0 \\ u_i^n \ge& -\frac{\Delta x}{\Delta t} \end{aligned}$$

$$\begin{aligned} \frac{\partial f_{i+1/2}^n}{\partial u_{i+1}^n} =& -\frac{1}{2}\frac{\Delta x}{\Delta t} + \frac{1}{2}(u_{i+1}^n) \le 0 \\ u_{i+1}^n \le& \frac{\Delta x}{\Delta t} \end{aligned}$$

So,

$$\boxed{|u_i^n| \le \frac{\Delta x}{\Delta t}}$$

(g) Using $C = u_i^n \frac{\Delta t}{\Delta x}$, rewriting equation (2) for linear advection gives:

$$u_i^{n+1} = \frac{1}{2}\left(1 + u_i^n \frac{\Delta t}{\Delta x}\right) u_{i-1}^n + \frac{1}{2}\left(1 - u_i^n \frac{\Delta t}{\Delta x}\right) u_{i+1}^n$$

$$u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) + \frac{1}{2}\frac{\Delta t}{\Delta x}(u_i^n u_{i-1}^n - u_i^n u_{i+1}^n)$$

Then, for Burger's equation (LF in primitive form):

$$u_i^{n+1} = \frac{1}{2}(u_{i-1}^n + u_{i+1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\frac{1}{2}(u_{i+1}^n)^2 - \frac{1}{2}(u_{i-1}^n)^2\right)$$

Massage the primitive form to give two "symmetric" flux terms. Consider Lax-Friedrichs scheme for the linear advection equation:

$$u_i^{n+1} = \frac{1}{2}(1+C)u_{i-1}^n + \frac{1}{2}(1-C)u_{i+1}^n$$

$$\begin{aligned}
u_i^{n+1} &= u_i^n - u_i^n + \frac{1}{2}u_{i-1}^n + \frac{1}{2}u_{i+1} - \frac{\Delta t}{2\Delta x}(au_{i+1}^n - au_{i-1}^n) \\
&= u_i^n + \frac{1}{2}(u_{i+1}^n - u_i^n) - \frac{1}{2}(u_i^n - u_{i-1}^n) - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}\left(f(u_{i+1}^n) + f(u_i^n) - f(u_i^n) - f(u_{i-1}^n)\right)\right] \\
&= u_i^n - \frac{\Delta t}{\Delta x}\Big[\frac{1}{2}\frac{\Delta x}{\Delta t}(u_i^n - u_{i-1}^n) + \frac{1}{2}\frac{\Delta x}{\Delta t}(u_i^n - u_{i+1}^n) + \frac{1}{2}(f(u_{i+1}^n) + f(u_i^n)) \\
&\qquad - \frac{1}{2}(f(u_i^n) + f(u_{i-1}^n))\Big] \\
&= u_i^n - \frac{\Delta t}{\Delta x}\Bigg\{\left[\frac{1}{2}\frac{\Delta x}{\Delta t}(u_i^n - u_{i+1}^n) + \frac{1}{2}(f(u_{i+1}^n) + f(u_i^n))\right] - \\
&\qquad \left[\frac{1}{2}\frac{\Delta x}{\Delta t}(u_{i-1}^n - u_i^n) + \frac{1}{2}(f(u_i^n) + f(u_{i-1}^n))\right]\Bigg\} \\
&= u_i^n - \frac{\Delta t}{\Delta x}(f_{i+1/2}^n - f_{i-1/2}^n)
\end{aligned}$$

So,

$$f_{i+1/2}^{LF} = \frac{1}{2}\frac{\Delta x}{\Delta t}(u_i^n - u_{i+1}^n) + \frac{1}{2}(f(u_i^n) + f(u_{i+1}^n))$$

For conservative form (preferable):

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^{LF} - f_{i-1/2}^{LF}\right)$$

Consider:

$$\frac{\partial H}{\partial u_{i-1}^n} = \frac{1}{2} + \frac{1}{2}\frac{\Delta t}{\Delta x}u_{i-1}^n \geq 0 \quad \Rightarrow \quad u_{i-1}^n \geq -\frac{\Delta x}{\Delta t}$$

$$\frac{\partial H}{\partial u_{i+1}^n} = \frac{1}{2} - \frac{1}{2}\frac{\Delta t}{\Delta x}u_{i+1}^n \geq 0 \quad \Rightarrow \quad u_{i+1}^n \leq \frac{\Delta x}{\Delta t}$$

So,

$$\boxed{-\frac{\Delta x}{\Delta t} \leq u_i^n \leq \frac{\Delta x}{\Delta t}}$$

(h) Conservative schemes only.

(i) No. Godunov's theorem states that monotone methods are at most first order accurate. LW scheme is a second-order method.

(j) Flux limiting can be used to give us a high-resolution total variation diminishing (TVD) method, through using a non-linear combination of a high-order scheme and a low-order monotone scheme. Higher-order methods are used everywhere except near discontinuities, where first-order method is switched on. This allows us to achieve a second-order accurate oscillation-free solution.

$$\boldsymbol{f}^{TVD}_{i+1/2} = \boldsymbol{f}^{LO}_{i+1/2} + \phi_{i+1/2}(\boldsymbol{f}^{HI}_{i+1/2} - \boldsymbol{f}^{LO}_{i+1/2})$$

where $\phi_{i+1/2} = \phi_{i+1/2}(r)$ is the limiter and r is the slope ratio.

$$r = \begin{cases} \frac{\Delta_{i-1/2}}{\Delta_{i+1/2}} & , a_i \geq 0 \\ \frac{\Delta_{i+1/2}}{\Delta_{i-1/2}} & , a_i \leq 0 \end{cases}$$

An example of a limiter function is Van-Leer or Minbee.

9.2 Q2

2. Same question as 2022-23 (Mock) Q2. Link

9.3 Q3

3. Same question as 2023-24 (Mock) Q3. Link

10 Appendix

10.1 Method of Characteristic

Method of Characteristic (PDE to ODE): the idea is that for a PDE in characteristic form, we can identify lines along which characteristic variables, $\boldsymbol{\nu}$, are constant i.e. $\frac{\mathrm{d}\nu}{\mathrm{d}s} = 0$, where $s = t$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda$ for linear advection equation.

Consider the characteristic variable form of a PDE (only if Jacobian is diagonalisable):

$$\frac{\partial \boldsymbol{\nu}}{\partial t} + \boldsymbol{\Lambda}\frac{\partial \boldsymbol{\nu}}{\partial x} = 0 \quad \text{where} \quad d\boldsymbol{\nu} = \boldsymbol{C}^{-1} d\boldsymbol{u} \quad \text{along} \quad dx = \lambda_i dt$$

The characteristic variable decouple the system of equations into m linear advection PDEs:

$$\frac{\partial \nu_i}{\partial t} + \lambda_i \frac{\partial \nu_i}{\partial x} = 0$$

with m characteristic curves satisfying m ODEs: $\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_i$. For a given initial condition, the solution is:

$$\nu_i = \nu_i^{(0)}(x - \lambda_i t)$$

which is just the advection of initial data. Transform back from char. variable to original variable: $\boldsymbol{u} = \boldsymbol{C}\boldsymbol{\nu}$, where $\boldsymbol{C}$ is the matrix of right eigenvectors and $\boldsymbol{C}^{-1}$ is the matrix of left eigenvectors.

$$\boldsymbol{u}(x,t) = \nu_i \cdot \boldsymbol{C}^{(i)} = \nu_1 \boldsymbol{C}^{(1)} + \nu_2 + \boldsymbol{C}^{(2)} + ...$$

i.e. given a point (x,t) in the $x - t$ plane, the solution $u_i(x,t)$ at this point depends only on the initial data at the m points $x_0^{(i)} = x - \lambda_i t$. These are the intersection of the characteristic speeds with the x-axis. The solution for u can be seen as the superposition of m waves, each of which is advected independently without change in shape.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix} = \nu_1 \begin{pmatrix} k_1^{(1)} \\ k_2^{(1)} \\ k_3^{(1)} \\ \vdots \end{pmatrix} + \nu_2 \begin{pmatrix} k_1^{(2)} \\ k_2^{(2)} \\ k_3^{(3)} \\ \vdots \end{pmatrix} + \nu_3 \begin{pmatrix} k_1^{(3)} \\ k_2^{(3)} \\ k_3^{(3)} \\ \vdots \end{pmatrix} + ...$$

Essentially, the solution is a linear combination of the right eigenvectors.

For real eigenvalues λ_i and the corresponding right eigenvectors $\boldsymbol{K}^{(i)}$. The characteristic speed defines a characteristic field, the λ_i-field.

λ_i-field is linearly degenerate if:

$$\boldsymbol{\nabla}\lambda_i(\boldsymbol{U}) \cdot \boldsymbol{K}^{(i)}(\boldsymbol{U}) = 0$$

λ_i-field is genuinely nonlinear if:

$$\boldsymbol{\nabla}\lambda_i(\boldsymbol{U}) \cdot \boldsymbol{K}^{(i)}(\boldsymbol{U}) \neq 0$$

For Euler equations,

$$d\boldsymbol{\nu} = \begin{pmatrix} d\nu_- \\ d\nu_0 \\ d\nu_+ \end{pmatrix} = \boldsymbol{C}^{-1} d\boldsymbol{u} = \begin{pmatrix} dv - \frac{dp}{\rho c_s} \\ d\rho - \frac{dp}{c_s^2} \\ dv + \frac{dp}{\rho c_s} \end{pmatrix} \quad \text{along} \quad \begin{matrix} dx = (v - c_s)dt \\ dx = v \\ dx = (v + c_s)dt \end{matrix}$$

Along characteristic lines, the characteristic variable is constant, and the constants are called Riemann invariants. Hence, the Riemann invariants are basically:

$$d\boldsymbol{\nu} = 0$$

For ν_-,

Rarefaction: Across a rarefaction wave, entropy is constant. This means:

$$\frac{p}{\rho^\gamma} = \text{constant}$$

We know for ideal gas:

$$p = (\gamma - 1)\rho\varepsilon \quad \text{and} \quad c_s^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho}\bigg|_s = \frac{\gamma p}{\rho}$$

10.2 Derivation of Rankine-Hugoniot Conditions

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{u}) = 0$$

Integrate both sides w.r.t. x,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_a^b \boldsymbol{u}\, dx = -[\boldsymbol{f}(\boldsymbol{u}(t,b)) - \boldsymbol{f}(\boldsymbol{u}(t,a))]$$

Let's say there is a single discontinuity $x_s \in [a,b]$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^{x_s^-(t)} \boldsymbol{u}\, dx + \int_{x_s^+(t)}^b \boldsymbol{u}\, dx \right) = -[\boldsymbol{f}(\boldsymbol{u}(t,b)) - \boldsymbol{f}(\boldsymbol{u}(t,a))]$$

Using Leibniz's integral rule,

$$\boldsymbol{u}(t,x_s^-)\frac{\partial x_s^-}{\partial t} + \cancel{\int_a^{x_s^-(t)} \frac{\partial \boldsymbol{u}}{\partial t}}\, dx - \boldsymbol{u}(t,x_s^+)\frac{\partial x_s^+}{\partial t} + \cancel{\int_{x_s^+(t)}^b \frac{\partial \boldsymbol{u}}{\partial t}}\, dx = -[\boldsymbol{f}(\boldsymbol{u}(t,b)) - \boldsymbol{f}(\boldsymbol{u}(t,a))]$$

Take limit, $a \to x_s^-$ and $b \to x_s^+$, integral vanish and $S = \frac{\partial x_s^-}{\partial t} = \frac{\partial x_s^+}{\partial t}$

$$\boldsymbol{u}(t,x_s^-)S - \boldsymbol{u}(t,x_s^+)S = -[\boldsymbol{f}(\boldsymbol{u}(t,x_s^+)) - \boldsymbol{f}(\boldsymbol{u}(t,x_s^-))]$$

$$\Rightarrow \quad S(\boldsymbol{u}_R - \boldsymbol{u}_L) = \boldsymbol{f}(\boldsymbol{u}_R) - \boldsymbol{f}(\boldsymbol{u}_L)$$

10.3 Conservation Form of Difference Schemes

1. **Lax-Friedrichs Scheme**

$$u_i^{n+1} = \frac{1}{2}(1+c)u_{i-1}^n + \frac{1}{2}(1-c)u_{i+1}^n$$

We can view LF scheme as an integral average within cell i,

$$u_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}\left(x, \frac{1}{2}\Delta t\right) dx$$

where,

$$\tilde{u}(x/t) = \begin{cases} u_{i-1}^n & , \text{ if } x/t < a \\ u_{i+1}^n & , \text{ if } x/t > a \end{cases}$$

$$\begin{aligned}
u_i^{n+1} &= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{u}\left(x, \frac{1}{2}\Delta t\right) dx \\
&= \frac{1}{\Delta x}\int_{x_{i-1/2}}^{a\Delta t/2} u_{i-1}^n \, dx + \frac{1}{\Delta x}\int_{a\Delta t/2}^{x_{i+1/2}} u_{i+1}^n \, dx \\
&= \frac{a\Delta t/2 - x_{i-1/2}}{\Delta x} u_{i-1}^n + \frac{x_{i+1/2} - a\Delta t/2}{\Delta x} u_{i+1}^n \\
&= \left(\frac{1}{2}c + \frac{1}{2}\right) u_{i-1}^n + \left(\frac{1}{2} - \frac{1}{2}c\right) u_{i+1}^n \\
&= \frac{1}{2}(1+c)u_{i-1}^n + \frac{1}{2}(1-c)u_{i+1}^n
\end{aligned}$$

The Lax-Friedrichs solution at cell i is a weighted average of the solution of the Riemann problem with the left and right neighbour states as data, at time $t = \frac{1}{2}\Delta t$. We can also say it is upwind bias since the upwind term always has the larger weight.

Generalise to non-linear system of conservation laws:

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$$

$$\text{Lax-Friedrichs:} \quad \boldsymbol{U}_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{\boldsymbol{U}}\left(x, \frac{1}{2}\Delta t\right) dx$$

Integrating the conservation law within control volume $[-\frac{1}{2}\Delta x, \frac{1}{2}\Delta x] \times [0, \frac{1}{2}\Delta t]$,

$$\begin{aligned}
\int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \tilde{\boldsymbol{U}}(x, \frac{1}{2}\Delta t)\, dx &= \int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \tilde{\boldsymbol{U}}(x, 0)\, dx + \int_0^{\frac{1}{2}\Delta t} \boldsymbol{F}(\tilde{\boldsymbol{U}}(-\frac{1}{2}\Delta x, t))\, dt - \int_0^{\frac{1}{2}\Delta t} \boldsymbol{F}(\tilde{\boldsymbol{U}}(\frac{1}{2}\Delta x, t))\, dt \\
&= \frac{1}{2}\Delta x\left(\boldsymbol{U}_{i+1}^n + \boldsymbol{U}_{i-1}^n\right) + \frac{1}{2}\Delta t \boldsymbol{F}_{i-1/2} - \frac{1}{2}\Delta t \boldsymbol{F}_{i+1/2}
\end{aligned}$$

Then,

$$\boldsymbol{U}_i^{n+1} = \frac{1}{2}\left(\boldsymbol{U}_{i+1}^n + \boldsymbol{U}_{i-1}^n\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i-1/2} - \boldsymbol{F}_{i+1/2}\right)$$

Putting it into the conservation update form:

$$\boldsymbol{U}_i^{n+1} = \boldsymbol{U}_i^n + \frac{\Delta t}{\Delta x}\left(\boldsymbol{F}_{i-1/2}^{LF} - \boldsymbol{F}_{i+1/2}^{LF}\right)$$

algebraic manipulation gives us the expression for the Lax-Friedrichs flux:

$$\boxed{\boldsymbol{F}_{i+1/2}^{LF} = \frac{1}{2}(\boldsymbol{F}_i^n + \boldsymbol{F}_{i+1}^n) + \frac{1}{2}\frac{\Delta x}{\Delta t}(\boldsymbol{U}_i^n - \boldsymbol{U}_{i+1}^n)}$$

2. **Lax-Wendroff Scheme**

$$u_i^{n+1} = \frac{1}{2}c(1+c)u_{i-1}^n + (1-c^2)u_i^n - \frac{1}{2}c(1-c)u_{i+1}^n$$

Writing it in conservative form for linear advection equation we identify the intercell numerical flux:

$$\begin{aligned}
u_i^{n+1} &= u_i^n - c\frac{\Delta t}{\Delta x}(au_i^n) + \frac{1}{2}\frac{\Delta t}{\Delta x}(1+c)(au_{i-1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}(1-c)(au_{i+1}^n) \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left[c(au_i^n) - \frac{1}{2}(1+c)(au_{i-1}^n) + \frac{1}{2}(1-c)(au_{i+1}^n)\right] \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1-c}{2}(au_{i+1}^n) + \frac{1+c}{2}(au_i^n) - \frac{1-c}{2}(au_i^n) - \frac{1+c}{2}(au_{i-1}^n)\right]
\end{aligned}$$

Hence,

$$\boxed{\boldsymbol{f}_{i+1/2}^{LW} = \frac{1+c}{2}\boldsymbol{f}_i^n + \frac{1-c}{2}\boldsymbol{f}_{i+1}^n}$$

It is a weighted average of fluxes of the left and right of the interface.

It can also be obtained from:

$$\boldsymbol{f}_{i+1/2}^{LW} = f(\boldsymbol{u}_{i+1/2}^{n+1/2}) \quad \text{where} \quad \boldsymbol{u}_{i+1/2}^{n+1/2} = \frac{1+c}{2}u_i^n + \frac{1-c}{2}u_{i+1}^n$$

for which $\boldsymbol{u}_{i+1/2}^{n+1/2}$ is a half-time step Lax-Friedrichs update!

For non-scalar conservation laws, we have the integral formulation instead:

$$f_{i+1/2} = f(u_{i+1/2}^{n+1/2}) \quad \text{where} \quad u_{i+1/2}^{n+1/2} = \frac{1}{\Delta x}\int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} u_{i+1/2}(x, \frac{1}{2}\Delta t)\,\Delta x$$

3. **Warming-Beam** ($a > 0$)

$$u_i^{n+1} = \frac{1}{2}c(c-1)u_{i-2}^n + c(2-c)u_{i-1}^n + \frac{1}{2}(c-1)(c-2)u_i^n$$

with CFL condition: $0 \leq |C| \leq 2$. It is a fully one-sided scheme where the stencil is taken from only the upwind direction. The enlarged stability range means one may advance in time with a larger time step Δt, boosting efficiency.

Similarly, we can attempt to write it in conservative form for linear advection equation to identify the intercell numerical flux:

$$\begin{aligned}
u_i^{n+1} &= \frac{1}{2}c(c-1)u_{i-2}^n + c(2-c)u_{i-1}^n + \frac{1}{2}(c-1)(c-2)u_i^n \\
&= \frac{1}{2}\frac{\Delta t}{\Delta x}(c-1)f_{i-2}^n + \frac{\Delta t}{\Delta x}(2-c)f_{i-1}^n + \frac{1}{2}(c^2-3c+2)u_i^n \\
&= u_i^n + \frac{1}{2}\frac{\Delta t}{\Delta x}(c-1)f_{i-2}^n + \frac{\Delta t}{\Delta x}(2-c)f_{i-1}^n + \frac{1}{2}\frac{\Delta t}{\Delta x}(c-3)f_i^n \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left[-\frac{1}{2}(c-1)f_{i-2}^n - (2-c)f_{i-1}^n - \frac{1}{2}(c-3)f_i^n\right] \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(3-c)f_i^n - \frac{1}{2}(4-2c)f_{i-1}^n - \frac{1}{2}(c-1)f_{i-2}^n\right] \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(3-c)f_i^n - \frac{1}{2}(3-c+1-c)f_{i-1}^n - \frac{1}{2}(c-1)f_{i-2}^n\right] \\
&= u_i^n - \frac{\Delta t}{\Delta x}\left[\frac{1}{2}(3-c)f_i^n + \frac{1}{2}(c-1)f_{i-1}^n - \frac{1}{2}(3-c)f_{i-1}^n - \frac{1}{2}(c-1)f_{i-2}^n\right]
\end{aligned}$$

Hence,

$$\boxed{\boldsymbol{f}_{i+1/2}^{WB} = \frac{1}{2}(3-c)\boldsymbol{f}_i^n + \frac{1}{2}(c-1)f_{i-1}^n}$$

10.4 Modified Equation

1. **First-order upwind Method**

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

is actually equivalent to:

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2} + a\frac{\partial u}{\partial x} - a\frac{\partial^2 u}{\partial x^2}\frac{\Delta x}{2} + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2) = 0$$

Noting that:

$$(\partial_t - a\partial_x)(\partial_t + a\partial_x)u = 0$$

Then,

$$(\partial_{tt} - a^2\partial_{xx})u = 0$$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \left(\frac{a\Delta x}{2} - \frac{a^2\Delta t}{2}\right)\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{1}{2}a\Delta x(1 - |c|)\frac{\partial^2 u}{\partial x^2}$$

The numerical viscosity is:

$$\alpha_{UW} = \frac{1}{2}a\Delta x(1 - |c|)$$

2. **Lax-Friedrichs Scheme**

$$u_i^{n+1} = \frac{1}{2}(1+c)u_{i-1}^n + \frac{1}{2}(1-c)u_{i+1}^n$$

$$\frac{1}{2}(u_i^{n+1} - u_{i+1}^n) + \frac{1}{2}(u_i^{n+1} - u_{i-1}^n) + \frac{1}{2}c(u_{i+1}^n - u_{i-1}^n) = 0$$

$$\begin{aligned}&\frac{1}{2}\left(u + u_t\Delta t + u_{tt}\frac{\Delta t^2}{2} - u - u_x\Delta x - u_{xx}\frac{\Delta x^2}{2}\right)\\ &+\frac{1}{2}\left(u + u_t\Delta t + u_{tt}\frac{\Delta t^2}{2} - u + u_x\Delta x - u_{xx}\frac{\Delta x^2}{2}\right)\\ &+\frac{1}{2}c\left(u + u_x\Delta x + u_{xx}\frac{\Delta x^2}{2} - u + u_x\Delta x - u_{xx}\frac{\Delta x^2}{2}\right) = 0\end{aligned}$$

$$\frac{1}{2}\left(u_t\Delta t + u_{tt}\frac{\Delta t^2}{2} - u_x\Delta x - u_{xx}\frac{\Delta x^2}{2}\right) + \frac{1}{2}\left(u_t\Delta t + u_{tt}\frac{\Delta t^2}{2} + u_x\Delta x - u_{xx}\frac{\Delta x^2}{2}\right)$$
$$+\frac{1}{2}c\left(2u_x\Delta x\right) = 0$$

$$\frac{1}{2}\left(2u_t\Delta t + u_{tt}\Delta t^2 - u_{xx}\Delta x^2\right) + c\left(u_x\Delta x\right) = 0$$

$$u_t\Delta t + \frac{1}{2}a^2 u_{xx}\Delta t^2 - \frac{1}{2}u_{xx}\Delta x^2 + au_x\Delta t = 0$$

$$\begin{aligned} u_t + au_x &= \frac{1}{2}\left(\frac{\Delta x^2}{\Delta t} - a^2\Delta t\right)u_{xx} \\ &= \frac{1}{2}\Delta x\left(\frac{\Delta x}{\Delta t} - a^2\frac{\Delta t}{\Delta x}\right)u_{xx} \\ &= \frac{1}{2}\Delta x(\frac{a}{c} - ac)u_{xx} \\ &= \frac{\Delta x a}{2c}(1-c^2)u_{xx} \end{aligned}$$

The numerical viscosity is:

$$\alpha_{LF} = \frac{a\Delta x}{2c}(1-c^2)$$

Let's take their ratio and take $0 \leq c \leq 1$,

$$\frac{\alpha_{LF}}{\alpha_{UW}} = \frac{\frac{a\Delta x}{2c}(1-c^2)}{\frac{1}{2}a\Delta x(1-|c|)} = \frac{1+c}{c}$$

$$2 \leq \frac{1+c}{c} \leq \infty$$

10.5 Monotone method

For a method of the form:

$$u_i^{n+1} = H(u_{i-k_L}^n, ..., u_{i+k_R}^n) = \sum_{k=-k_L}^{k_R} b_k u_{i+k}^n$$

It is monotone when all coefficients are positive or zero $b_k \geq 0, \forall\, k$, in other words:

$$\frac{\partial H}{\partial u_j^n} \geq 0 \quad , \forall\, j$$

That is, **H is a non-decreasing function of each of its argument**.

The definition of a monotone scheme is actually equivalent to the following property:

$$\text{if} \quad v_i^n \geq u_i^n \,\forall\, i \quad \text{then} \quad v_i^{n+1} \geq u_i^{n+1}$$

This property is the discrete version of the following property of the exact solution of the conservation law: if two initial data functions satisfy $v_0(x) > u_0(x)\,\forall x$, then their corresponding solutions satisfy $v(x,t) > u(x,t)\,\forall t$.

Hence, **monotone schemes mimic a basic property of exact sols**. of conservation laws.

Monotone methods do not form spurious oscillations. Given the data set $\{u_i^n\}$, if the solution set $\{u_i^{n+1}\}$ if obtained with monotone method, then

$$\max_i\{u_i^{n+1}\} \leq \max_i\{u_i^n\} \quad \text{and} \quad \min_i\{u_i^{n+1}\} \geq \min_i\{u_i^n\}$$

Proof: Define $v_i^n = \max_j\{u_j^n\} = \text{constant}$, then evolving it gives $v_i^{n+1} = v_i^n$. As $v_i^n \geq u_i^n$, then $v_i^{n+1} = v_i^n \geq u_i^{n+1}$. Therefore, $\max_i\{u_i^{n+1}\} \leq \max_i\{u_i^n\}$. An obvious consequence is:

$$\max_i\{u_i^n\} \leq ... \leq \max_i\{u_i^0\} \quad \text{and} \quad \min_i\{u_i^n\} \geq ... \geq \max_i\{u_i^0\}$$

Hence, no new extrema are created, and thus spurious oscillations do not appear. In numerical solutions computed with monotone methods, minima increase and maxima decrease as time evolves. This results in clipping of extrema, which is in fact a disadvantage of monotone methods.

Another consequence is: $\min_j\{u_j^n\} \leq \min_j\{u_j^{n+1}\} \leq u_i^{n+1} \leq \max_j\{u_j^{n+1}\} \leq \max_j\{u_j^n\}$. This says that the solution at any point i is bounded by the minimum and maximum of the data.

OR in terms of fluxes:

$$\frac{\partial f_{i+1/2}}{\partial u_i^n} \geq 0 \quad \text{and} \quad \frac{\partial f_{i+1/2}}{\partial u_{i+1}^n} \leq 0$$

In other words, the numerical flux $f_{i+1/2}(u_i^n, u_{i+1}^n)$ is an increasing function of its first argument and a decreasing function of its second argument.

Example: applying it to LF flux:

$$f_{i+1/2}^{LF} = \frac{1}{2}[f(u_i^n) + f(u_{i+1}^n)] + \frac{1}{2}\frac{\Delta x}{\Delta t}(u_i^n - u_{i+1}^n)$$

$$\frac{\partial f_{i+1/2}^{LF}}{\partial u_i^n} = \frac{1}{2}f' + \frac{1}{2}\frac{\Delta x}{\Delta t} \geq 0$$

$$f' \geq -\frac{\Delta x}{\Delta t}$$

Likewise,

$$\frac{\partial f_{i+1/2}^{LF}}{\partial u_{i+1}^n} = \frac{1}{2}f' - \frac{1}{2}\frac{\Delta x}{\Delta t} \leq 0$$

$$f' \leq \frac{\Delta x}{\Delta t}$$

Overall, we have the condition of:

$$\frac{-\Delta x}{\Delta t} \leq f' \leq \frac{\Delta x}{\Delta t}$$

$$-1 \leq \frac{f'\Delta t}{\Delta x} \leq 1$$

Notice that $\lambda(u) = \frac{\partial f}{\partial u}$ are the characteristic speeds, and we will take f' to be the maximum eigenvalue of the system to ensure the condition is definitely satisfied. We recover the CFL condition for stable numerical method.

10.6 Degenerate Waves

For example, in 2D Euler equations, the primitive variable form is:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v_x\\ \rho v_y\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho v_x\\ \rho v_x^2+p\\ \rho v_x v_y\\ (E+p)v_x\end{pmatrix}=0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\begin{pmatrix}\rho\\ v_x\\ v_y\\ p\end{pmatrix}+\begin{pmatrix}v_x & \rho & 0 & 0\\ 0 & v_x & 0 & \frac{1}{\rho}\\ 0 & 0 & v_x & 0\\ 0 & \rho c_s^2 & 0 & v_x\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ v_x\\ v_y\\ p\end{pmatrix}=0$$

The eigenvalues are:

$$\lambda_1 = v_x - c_s \quad , \quad \lambda_2=\lambda_3=v_x \quad , \quad \lambda_4 = v_x + c_s$$

We have 4 real eigenvalues, but they are not all distinct. The system is hyperbolic, but not strictly hyperbolic.

There could still be 4 distinct waves in the solution. The multiple waves with the eigenvalues are called degenerate waves. Degenerate waves are indistinguishable in the solution, but have different mathematical properties.

Characteristic variables:

$$d\boldsymbol{v}=\begin{pmatrix}dv_0\\ dv_+\\ dv_-\\ dv_{\text{sh}}\end{pmatrix}=\begin{pmatrix}d\rho-\frac{dp}{c_s^2}\\ dv_x+\frac{dp}{\rho c_s}\\ dv_x-\frac{dp}{\rho c_s}\\ dv_y\end{pmatrix} \quad \text{along} \quad \begin{cases}dx=v_x dt\\ dx=(v_x+c_s)dt\\ dx=(v_x-c_s)dt\\ dx=v_x dt\end{cases}$$

- The fourth characteristic variable is associated with a **shear wave**.
- Shear wave moves at the same speed as the contact discontinuity.
- A wave where the velocity is moving perpendicular to the wave.
- Across a shear wave, only the transverse velocity i.e. v_y jumps.

Consequences for Riemann problem: transverse velocity jump effectively decouples from the other quantities in the characteristic variables, so the shear wave (coincides with contact discontinuity) is the only characteristic line across which the transverse velocity jumps.

$$\boldsymbol{w}_L^*=\begin{pmatrix}\rho_L^*\\ v_x^*\\ v_{y,L}\\ p^*\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\rho_R^*\\ v_x^*\\ v_{y,R}\\ p^*\end{pmatrix}$$

Vanishing waves:

- For complex systems, other waves may be degenerate.
- Also, the other waves can exist mathematically, but variables might have zero-height jump across the waves.

10.7 Recommended Readings

1. **Course Book**: Toro, E. Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction. in Riemann Solvers and Numerical Methods for Fluid Dynamics (2009). doi:10.1007/b79761. Link