Oscillations, Waves and Optics Summary Notes

By Shikang Ni

1 Oscillations

Simple harmonic oscillator:

$$F(t) = m\ddot{x} + b\dot{x} + kx \Rightarrow \frac{F}{m} = \ddot{x} + \gamma\dot{x} + \omega_0^2 x$$

where

$$\gamma = \frac{b}{m} \quad \text{and} \quad \omega_0^2 = \frac{k}{m}$$

1.1 Homogeneous equation (free oscillation)

Trial solution: $x = e^{pt}$,

$$p = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2}\left(1 \pm \sqrt{1 - 4Q^2}\right)$$

where Q is the quality factor.

Damping regimes:
Light damping: $\gamma < 2\omega_0$ or $Q > 0.5$
Critical damping: $\gamma = 2\omega_0$ or $Q = 0.5$
Heavy damping: $\gamma > 2\omega_0$ or $Q < 0.5$

1.1.1 Light damping ($\gamma < 2\omega_0$ or $Q > 0.5$)

Description: There will be oscillations, enveloped by a decaying exponential of the form $e^{-\gamma t/2}$.

$$x(t) = e^{-\gamma t/2}(A \sin \omega_f t + B \cos \omega_f t) = e^{-\gamma t/2} D \cos(\omega_f t + \phi)$$

$$\text{Free oscillation frequency: } \omega_f = \frac{1}{4}\sqrt{4\omega_0^2 - \gamma^2}$$

Q Factor

$$\boxed{Q = \frac{\omega_0}{\gamma} = \frac{\omega_0}{\Delta\omega}}$$

Q is the number of radians of phase elapsed ($\omega_0 t$) for amplitude to fall to $e^{-0.5}$ of original value / intensity or energy to fall by e^{-1}.

$$A(t) = A_0\, e^{-\gamma t/2} \quad \text{and} \quad I(t) = I_0\, e^{-\gamma t}$$

Amplitude decays with time constant $\tau = {}^2\!/\!_\gamma$ and intensity decays with $\tau = {}^1\!/\!_\gamma$.

1.1.2 Critical damping ($\gamma = 2\omega_0$ or $Q = 0.5$)

Description: there is no oscillations, and it has the fastest return to equilibrium.

$$x(t) = (C_1 + C_2\omega_0 t)e^{-\omega_0 t}$$

Most rapid approach to equilibrium with no overshooting. Time constant $\tau = \frac{1}{\omega_0}$. Example, analog speedometers.

1.1.3 Heavy damping ($\gamma > 2\omega_0$ or $Q < 0.5$)

Description: exponential decay to equilibrium with no oscillations, but slower than critical damping.

$$x(t) = C_1 e^{\mu_- t} + C_2 e^{\mu_+ t}$$

$$\mu_\pm = -\frac{1}{2}\left(\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right)$$

Check for overshooting by $x(t) \leq 0$.

1.2 Driven harmonic oscillator

Trial solution: $x = x_0 e^{i\omega t}$ where $x_0 = Ae^{i\phi}$ and $F = \Re[F_0 e^{i\omega t}]$

Amplitude response function:

$$R(\omega) = \frac{x_0}{F_0} = \frac{1}{m[(\omega_0 - \omega)^2 + i\gamma\omega]} \quad \text{or} \quad \frac{1}{-\omega^2 m + i\omega b + k}$$

$$= \frac{(\omega_0^2 - \omega^2) - i\gamma\omega}{m[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]}$$

Amplitude response:

$$|R| = \frac{1}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

$$arg(R) = \tan^{-1}\left[\frac{-\gamma\omega}{\omega_0^2 - \omega^2}\right]$$

To find amplitude resonance frequency, solve for ω in $\boldsymbol{\frac{d|R|}{d\omega} = 0}$ to obtain:

$$\omega_a = \omega_0\sqrt{1 - \frac{\gamma^2}{2\omega_0^2}} = \omega_0\sqrt{1 - \frac{1}{2Q^2}}$$

Velocity response

$$v_0 = i\omega x_0 = i\omega F_0 R(\omega)$$

So,

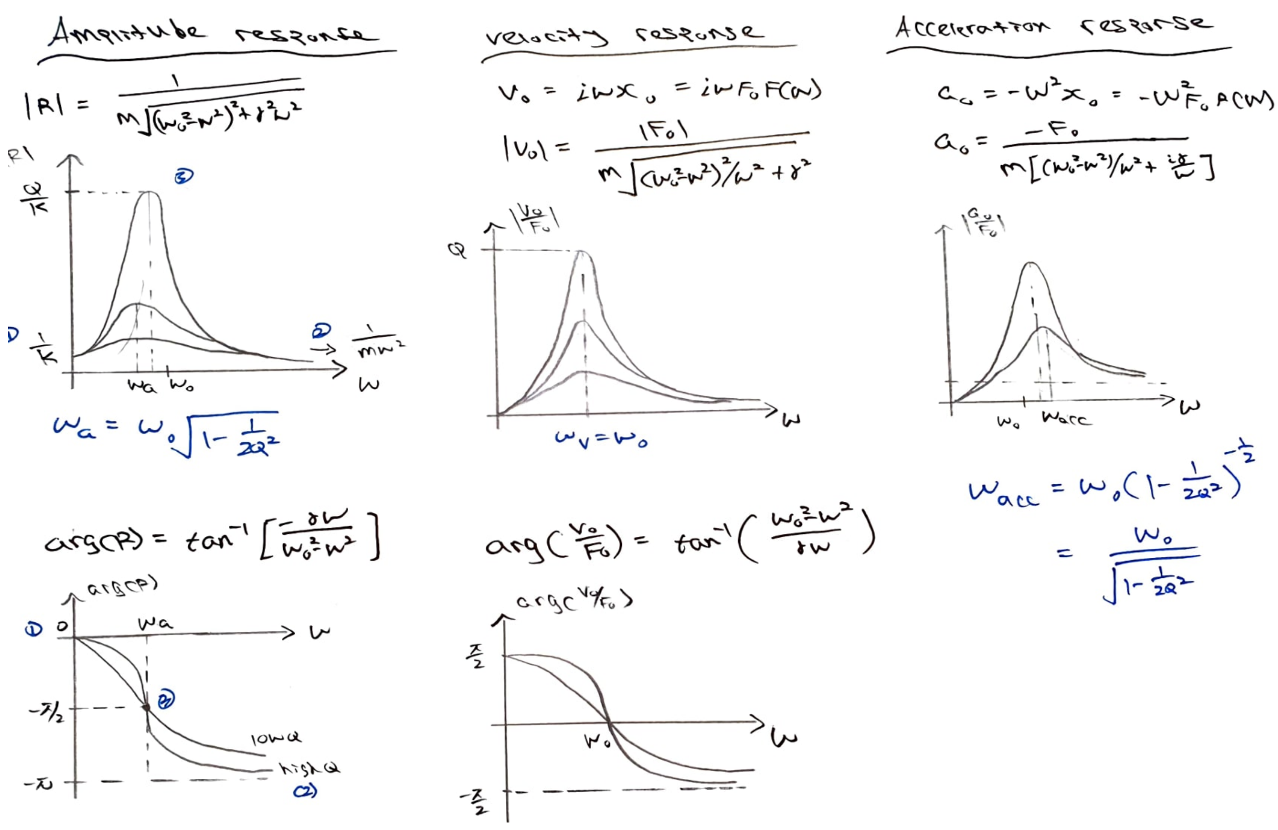
$$|v_0| = \frac{|F_0|}{m\sqrt{(\omega_0 - \omega)^2/\omega^2 + \gamma^2}}$$

$$arg\left(\frac{v_0}{F_0}\right) = \tan^{-1}\left(\frac{\omega_0^2 - \omega^2}{\gamma\omega}\right)$$

Acceleration response

$$a_0 = -\omega^2 x_0 = -\omega^2 F_0 R(\omega)$$

$$a_0 = -\frac{F_0}{m[(\omega_0 - \omega^2)/\omega^2 + i\gamma/\omega]}$$



1.3 Problem-solving tips

1. (2017 Tripos P2 B6) What happens to the motion of a body when a mass M is suddenly added to the system?

There will be a new equilibrium position of $l = -\frac{Mg}{k}$ below the original equilibrium position.
This change can be accounted for simply by updating the initial conditions.

$$\tilde{x}(t=0) = +\frac{Mg}{k} \quad \text{and} \quad \dot{\tilde{x}}(0) = 0$$

Then proceed as usual to solve the equation of motion using substitution $\tilde{x} = Ae^{\lambda t}$.

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2. Critical damping is always preferred in practice.

3.

2 Waves

2.1 Transverse Waves

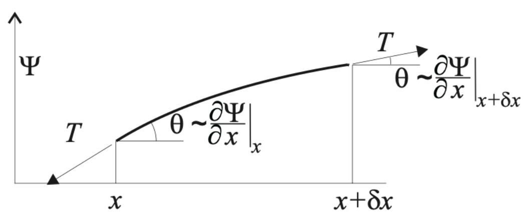
Transverse wave equation (non-dispersive):

$$\frac{\partial^2\psi}{\partial t^2} = v^2\nabla^2\psi$$

where $v = \sqrt{\frac{T}{\mu}}$ where T is tension force term and μ is the mass term.

Key properties: any waveform, non-dispersive, linear (principle of superposition)

Derivation of transverse wave equation



Assume uniform tension and gradient of string $\ll 1$. Find expression for 1) net tension force and use 2) Newton's second law.

Propagation of general disturbance

Waves of all frequencies travel at the same speed v, so does a disturbance (definition of non-dispersive - velocity does not depend on frequency).

$$\Psi(x,t) = \Psi(x \pm vt, 0) = f(x \pm vt)$$

Waves in various coordinate system

Spherical waves:

$$\text{Wave equation: } v^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) = \frac{\partial^2\Psi}{\partial t^2}$$

Guess:

$$\Psi(x,t) = \frac{f(r \pm vt)}{r}$$

Cylindrical waves:

$$\text{Wave equation: } v^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) = \frac{\partial^2\Psi}{\partial t^2}$$

Guess:

$$\Psi(x,t) \approx \frac{f(r \pm vt)}{\sqrt{r}} \quad \text{if} \quad r \gg \lambda$$

2.1.1 Polarisation

Relative phase, ϕ, is the most important parameter in polarisation.

Same amplitude + same phase $\Rightarrow$ linear polarisation.
Same amplitude + $\frac{\pi}{2}$ out of phase $\Rightarrow$ circular polarisation.
Arbitrary amplitude and phase $\Rightarrow$ elliptical polarisation.

Linear polarisation
Resolve the displacement into two perpendicular components in a plane perpendicular to the direction of propagation.

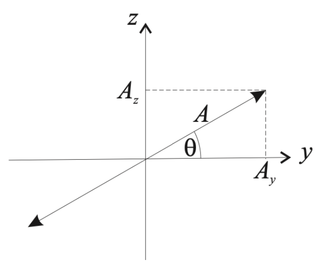


Figure 1: Linear polarisation

$$\begin{pmatrix} \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} A\cos\theta \\ A\sin\theta \cdot e^{in\pi} \end{pmatrix} = A\begin{pmatrix} \cos\theta \\ \pm\sin\theta \end{pmatrix}$$

Also, linear polarisation = sum of coherent left- and right-handed circularly polarised waves.

Circular polarisation
Convention for polarisation direction: look at wave coming towards you, clockwise motion of electric field vector is right-circular polarisation, vice versa.

$$\begin{pmatrix} \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} A \\ Ae^{i(2n+1)\pi/2} \end{pmatrix} = \begin{pmatrix} A \\ \pm iA \end{pmatrix}$$

+ for right-circular / - for left-circular.

Also, circular polarisation = sum of orthogonal linear polarisation with $\pi/2$ relative phase.

Elliptical polarisation

$$\Psi_y = A_y\cos(\omega t - kx)$$

$$\Psi_z = A_z\cos(\omega t - kx + \phi)$$

$$\Rightarrow \Psi_z = A_z\left(\frac{\Psi_y}{A_y}\cos\phi - \sqrt{1-\frac{\Psi_y^2}{A_y^2}}\sin\phi\right)$$

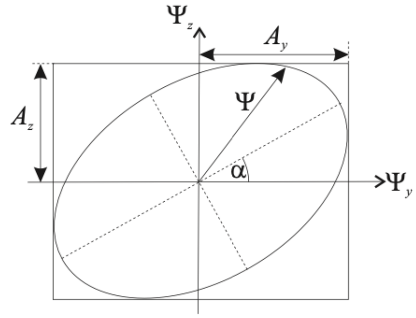


Figure 2: Linear polarisation

Can rearrange into ellipse equation.

$$\begin{pmatrix} \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} A \\ \pm iB \end{pmatrix}$$

2.1.2 Wave impedance

$$Z = \frac{\text{driving force}}{\text{velocity response}} = \frac{F}{v}$$

Example, free end of string:

$$Z = \frac{-T\sin\theta}{\dot{\psi}} = \frac{-T\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial t}} = \frac{(-T)\frac{df}{du}}{(-v)\frac{df}{du}} = \frac{T}{v}$$

Using $\psi = f(u) = f(x - vt)$ for moving in +x direction. Hence,

$$Z = \frac{T}{v} = \sqrt{T\rho} = \boldsymbol{\rho v}$$

2.1.3 Power in wave

$$\text{Power} = \text{force} \times \text{velocity} = Fv$$

$$\text{Mean power: } \langle P \rangle = \frac{1}{2}\Re[\mathbf{F}\mathbf{v}^*] = \frac{1}{2}\Re[Z]|\mathbf{v}|^2$$

Then, using $\mathbf{v} = \dot{\boldsymbol{\psi}} = i\omega\psi$,

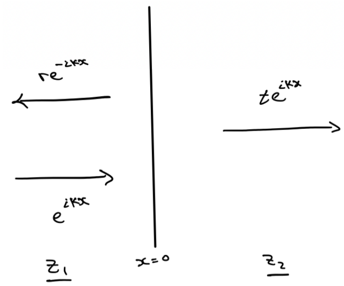
$$\langle P \rangle = \frac{1}{2}Z\omega^2 A_0^2 = (KE + PE) \times v$$

$$KE = \frac{1}{2}\rho\left(\frac{\partial\psi}{\partial t}\right)^2 = \frac{1}{2}\rho\omega^2 A_0^2 \quad \text{and} \quad PE = \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2 = \frac{1}{2}Tk^2 A_0^2$$

Note that $\rho\omega^2 = Tk^2$. After averaging over a complete wavelength, average energy per unit length is $\frac{1}{2}\rho\omega^2 A_0^2$.. Multiplying by v gives the mean power flow, which is the same expression as above.

2.1.4 Boundary questions

1. Problem set up:



2. Impose boundary conditions:

- Continuity of 'displacement', ψ.
- Continuity of 'force', $-T\frac{\partial\psi}{\partial x}$ or $Z\frac{\partial\psi}{\partial t}$.

  For sound,

- Continuity of acoustic excess pressure, ψ_p
- Continuity of particle velocity, $\dot{a}$

3. Solve:

Amplitude reflection coefficient:

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

Amplitude transmission coefficient:

$$t = \frac{2Z_1}{Z_1 + Z_2}$$

For pressure, voltage and E-field. Replace Z_1 with $\frac{1}{Z_1}$ and Z_2 with $\frac{1}{Z_2}$.

Force reflection coefficient:

$$r_f = \frac{Z_2 - Z_1}{Z_1 + Z_2}$$

Force transmission coefficient:

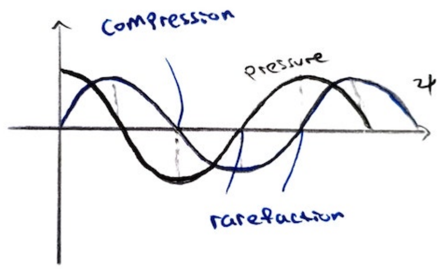
$$t_f = \frac{2Z_2}{Z_1 + Z_2}$$

Also the **power transmission coefficients** are:

$$R = rr^* \quad \text{and} \quad T = tt^* = 1 - R \quad \text{and} \quad R + T = 1$$

2.2 Longitudinal Waves

A series of rarefactions and compressions, no polarisation possible, maximum pressure at minimum displacement.



In gas,

$$\text{Wave equation: } \frac{\partial^2 a}{\partial x^2} = \frac{\rho}{\gamma p} \frac{\partial^2 a}{\partial t^2}$$

$$\text{*Speed of wave: } v = \sqrt{\frac{\gamma p}{\rho}}$$

$$\text{Acoustic impedance: } \boldsymbol{\mathcal{L}} = \boldsymbol{v\rho} = \sqrt{\gamma p \rho}$$

where γp can be generalised to some sort of modulus: elastic modulus (K), bulk modulus (B), Young's modulus (Y).

2.2.1 Derivation

Ingredient 1: reaction of gas to compression/rarefaction

An adiabatic process ($Q = 0$) as compression/rarefaction is so fast that heat cannot flow in and out of the gas before the next phase of pressure wave passes through.

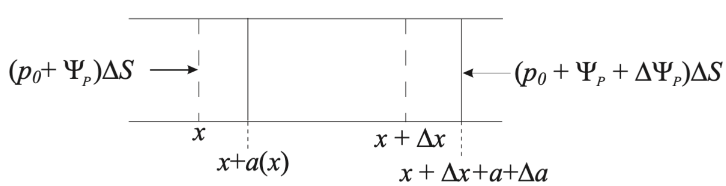
$$pV^\gamma = \text{constant}$$

$$dpV^\gamma + p\gamma V^{\gamma - 1} dV = 0$$

$$\Rightarrow dp = \boxed{\psi_p = -\gamma p \frac{\Delta V}{V}}$$

where $\gamma = \frac{C_V + R}{C_V}$.

Ingredient 2: strain in volume



$$\boxed{\frac{\Delta V}{V} = \frac{\Delta S \Delta a}{\Delta S \Delta x} = \frac{\partial a}{\partial x}}$$

where $\Delta a = \frac{\partial a}{\partial x} \cdot \Delta x$.

Ingredient 1 + ingredient 2:

$$\boxed{\psi_p = -\gamma p \frac{\partial a}{\partial x}}$$

$$\boxed{\frac{\partial \psi_p}{\partial x} = -\gamma p \frac{\partial^2 a}{\partial x^2}} - \underbrace{\gamma \frac{\partial p}{\partial x}\frac{\partial a}{\partial x}}_{\substack{\text{negligible} \\ \text{since } a \ll \lambda}}$$

Ingredient 3: stress

$$F_{net} = -\frac{\partial \psi_p}{\partial x} \Delta x \cdot \Delta S$$

$$\frac{m}{\Delta x \Delta S}\frac{\partial^2 a}{\partial t^2} = -\frac{\partial \psi_p}{\partial x} \Rightarrow \boxed{\rho \frac{\partial^2 a}{\partial t^2} = -\frac{\partial \psi_p}{\partial x}}$$

Combining the last two equations:

$$\rho \frac{\partial^2 a}{\partial t^2} = \gamma p \frac{\partial^2 a}{\partial x^2}$$

$$\therefore \boxed{\frac{\partial^2 a}{\partial t^2} = \frac{\gamma p}{\rho}\frac{\partial^2 a}{\partial x^2}}$$

2.2.2 Impedance matching

Make reflection zero by using an intermediate medium with the following properties:

1. d = $\frac{\lambda}{4}$
2. $Z_2 = \sqrt{Z_1 Z_3}$

Q1: Why does spectacle lens appear purple

A1: The lenses are coated with a medium whose refractive index is the geometric mean of those of the lens itself and air. If this layer is a quarter of a wavelength thick perfect transmission ensues. We choose green as the middle of the visible spectrum, and so this does not quite work for blue and red. Thus the reflected light seems purple.

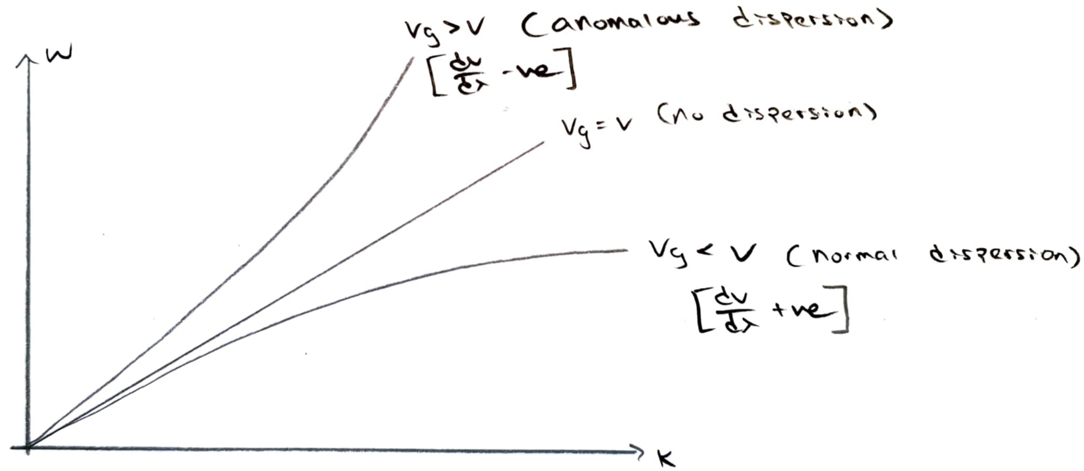
2.3 Dispersion

Dispersion relation is written in the form ω as a function of k:

$$\omega(k) = f(k)$$

For example, $\omega = v_{ph}k$.

2.3.1 Dispersion curve



2.4 Damped system

$$\frac{\partial^2\psi}{\partial t^2} + \underbrace{\Gamma\frac{\partial\psi}{\partial t}}_{\text{damping}} = v^2\frac{\partial^2\psi}{\partial x^2}$$

$$\boxed{\omega^2 - i\omega\Gamma = v^2k^2}$$

$$\Rightarrow k^2 = \frac{\omega^2}{v^2} - i\frac{\omega\Gamma}{v^2}$$

For light damping ($\Gamma \ll \omega$):

$$k \approx \frac{\Gamma}{2v}$$

For heavy damping ($\Gamma \gg \omega$):

$$k \approx \sqrt{\frac{\Gamma\omega}{2v^2}}$$

2.4.1 Wave impedance

$$Z = \frac{-T\Psi'}{\dot{\Psi}} = \frac{Tk}{\omega}$$

2.5 Water waves

Water moves in both longitudinal and transverse directions, with these oscillations in quadrature, forming ellipses.

Dispersion relation in deep water:

$$\omega^2 = \underbrace{gk}_{\text{gravity}} + \underbrace{\frac{\sigma k^3}{\rho}}_{\substack{\text{surface}\\ \text{tension}}}$$

1. **Ripple/capillary wave** - surface tension driven,

$$v_g \approx \frac{3}{2} v_\phi \quad \text{and} \quad v_\phi = \sqrt{\frac{\sigma k}{\rho}}$$

Phase speed and group speed decreases as wavelength increases. Anomalous dispersion. Individual waves appear to move backwards through the wavepacket.

2. **Gravity waves** - gravitational force driven,

$$v_g = \frac{1}{2} v_\phi$$

Phase speed and group speed increase as wavelength increases. Normal dispersion. Individual waves appear to move forwards through the wavepacket.

3. Shallow water gravity waves:

If wavelength λ exceeds water depth h, the wave motion is mainly longitudinal.

Example: Tsunami wave

2.6 Group and phase velocity

Phase velocity: the velocity that a pure frequency harmonic wave propagates.

Group velocity: velocity at which an interference feature of a wavepacket propagates.

Derivation 1

Consider two equal-amplitude waves propagating together with close frequencies ($k_1 \approx k_2$). We get beating.

$$\begin{aligned}\Psi &= \cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x) \\ &= 2\underbrace{\cos(\omega_+ t - k_+ x)}_{\text{carrier}}\underbrace{\cos(\omega_- t - k_- x)}_{\text{envelope}}\end{aligned}$$

where $\omega_+ = \frac{1}{2}(\omega_1 + \omega_2)$ and $\omega_- = \frac{1}{2}(\omega_1 - \omega_2)$. Likewise for k_+ and k_-.

It is made up of a high frequency wave travelling at speed which we call the **phase velocity**:

$$v_{ph} = \frac{\omega}{k}$$

This is modulated by a low frequency travelling envelope with speed we call the **group velocity**:

$$v_{grp} = \frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{d\omega}{dk}$$

Derivation 2

We can think of group velocity as the speed at which a "bump" in the wave travels.

A wave can be represented as a sum of many Fourier components (pure harmonic waves of different frequencies), with each having a phase of $(\omega t - kx)$. The bump arise when the constituent waves add up **in phase** at a point. This happens when:

$$\frac{d}{d\omega}(\omega t - kx)|_{\omega_0} = 0$$

where ω_0 is the angular frequency of the typical wave in the group.

$$t - x\left(\frac{dk}{d\omega}\right)_{\omega_0} = 0$$

$$\frac{x}{t} = v_{grp} = \frac{d\omega}{dk}$$

Derivation 3
Expanding the dispersion relation:

$$\omega(k) = \omega_0 + \frac{\partial \omega}{\partial k}\Big|_{k_0} k_1 + ...$$

The first order is the group velocity:

$$v_{grp} = \frac{\partial \omega}{\partial k}\Big|_{k_0}$$

For non-dispersive wave, the phase and group velocity is the same, so the envelope will propagate at the same speed as the carrier wave. For dispersive wave, wave crests will move relative to the envelope.

Relationship between spatial extent of group and range of wavevectors in a group:

$$\boxed{\Delta k \Delta x \approx 1}$$

Estimating rate of spreading of wave packet If a group contain wavevectors in a band of size $2\Delta k$ about k_0, there is a range of group velocities in the group between:

$$v_{g1} = \frac{\partial \omega}{\partial k}\Big|_{k_0 + \Delta k} \quad \text{and} \quad v_{g2} = \frac{\partial \omega}{\partial k}\Big|_{k_0 - \Delta k}$$

Expanding group velocity about k_0 to first order:

$$v_{g1} \approx v_g|_{k_0} + \underbrace{\frac{\partial^2 \omega}{\partial k^2}\Big|_{k_0}}_{\beta} \Delta k \quad \text{and} \quad v_{g2} \approx v_g - \beta \Delta k$$

The length of the wavepacket after time t is:

$$\Delta x \approx \Delta x_0 + (v_{g1} - v_{g2})t \approx \Delta x_0 + (2|\beta|\Delta k)t$$

$$\boxed{\therefore \Delta x \approx \Delta x_0 + \frac{2|\beta|}{\Delta x_0} t}$$

The second order term in the dispersion relation tells you about the broadening of the wave packet!

Just like how the first order term tell you about the group velocity.

Physical intuition to dispersive shape change: as longer wavelengths are faster, the waves will 'pile up' at the front of the wave packet, while the shorter wavelengths will be gathered at the back.

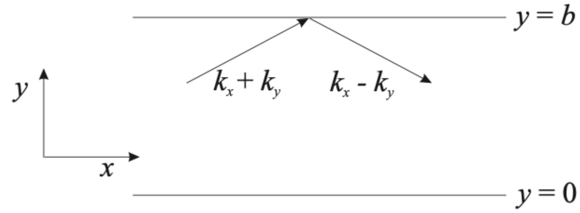
2.7 Guided Waves

Benefits:

- Allow us to send a single mode of oscillation at a given frequency ω_0 with sensible choice of the size parameters of the waveguide which determines the cutoff frequency.
- We do not get the $1/r^2$ decay of the transmitted wave so this allows weak signals to be sent over long distances. It also shield a signal from outside disturbances.

Applications: optical fibres, stethoscopes.

Set-up:



Consider two travelling waves with wavevector k, moving at angle $\pm\theta$ to x:

$$\Psi_A = Ae^{i(\omega t - k_x x - k_y y)}$$

$$\Psi_B = -Ae^{i(\omega t - k_x x + k_y y)}$$

$$\Psi = \Psi_A + \Psi_B = Ae^{i(\omega t - k_x x)}\left[e^{-ik_y y} - e^{-ik_y y}\right] = -2iA\sin(k_y y)e^{i(\omega t - k_x x)}$$

This is a wave travelling in the x-direction of wavelength $\frac{2\pi}{k_x}$, with amplitude modulated by a standing wave in the y-direction.

Boundary conditions in the y-direction gives discrete values for k_y:

$$k_y = \frac{n\pi}{b}$$

Dispersion relation

$$\boxed{\omega^2 = v^2\left(k_x^2 + \frac{n^2\pi^2}{b^2}\right)}$$

Allowing us to find phase velocity and group velocity:

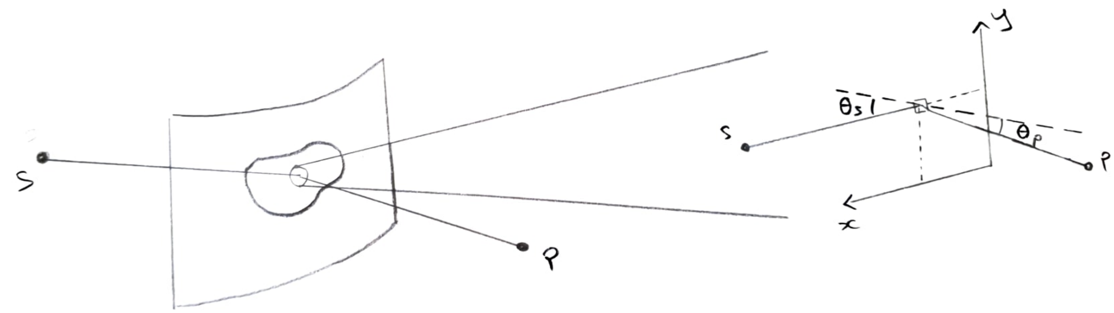
$$v_p = \frac{\omega}{k_x} \quad \text{and} \quad v_g = \frac{d\omega}{dk_x}$$

But noting the relationship between phase and group velocity allows us to find group velocity quickly:

$$\boxed{v_p v_g = v^2}$$

3 Diffraction

3.1 Fresnel-Kirchhoff Diffraction Integral



$$\Psi_p = \int\int \underbrace{-\frac{i}{\lambda}}_{1} \underbrace{h(x,y)}_{2} \underbrace{K(\theta_s,\theta_p)}_{3} \underbrace{\frac{a_s e^{iks}}{s} \cdot \frac{e^{ikr}}{r}}_{4} dxdy$$

1. Amplitude of wavelet relative to incoming wave
2. ***Aperture function**. $h(x,y) = 0$ or 1 depending on whether (x,y) is obstructed or open.
3. ***Obliquity/inclination factor** (Huygens-Fresnel theory): describes the fall-off in intensity of the wavelets with angle θ away from the forward direction.

$$K(\theta_s,\theta_p) = \frac{\cos_s + \cos_p}{2} \approx 1$$

4. ***Phase factors**:

$$\text{point S to } (x,y)\text{: } \frac{a_s e^{iks}}{s}$$

$$\text{*}\mathbf{point}\ (x,y)\ \mathbf{to\ P}\text{:} \frac{e^{ikr}}{r}$$

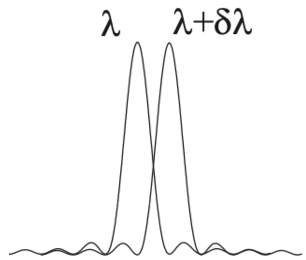
Huygens' principles: each point on a wavefront acts as a source of secondary wavelets which propagate, overlap, interfere, and thus carry the wavefront forward.

3.2 Resolution

3.2.1 Rayleigh criterion

Statement: images can be just resolved if the maximum for one pattern coincides with the minimum of the other pattern.

In other words, the separation of their peaks is equal to the full-width at half maximum.



Chromatic resolving power of grating is given by:

$$\boxed{\frac{\lambda}{\Delta\lambda} = mN = \frac{mW}{d}}$$

where m is the order and N is the number of slits, $N = \frac{\text{width}}{\text{slit spacing}} = \frac{W}{d}$.

Using wider gratings give narrower diffraction beams and observing at higher order increases angular separation of the patterns. Both makes it easier to revolve close wavelengths.

Intuitive approach:
1. Find location of peak, θ. E.g. $\theta = \sin^{-1}\frac{\lambda}{d}$

2. Find location of next zero, $\theta + \delta\theta$. E.g. $\theta + \delta\theta = \sin^{-1}[\frac{\lambda+\Delta\lambda}{d} + \frac{\lambda}{w}]$ where d is slit spacing and w is width of grating.

3.2.2 Airy disk

Fraunhofer diffraction through circular aperture would produce Airy disk (the first zero of the Bessel function of the first kind).

$$\text{angular resolution} = 1.22\frac{\lambda}{D}$$

$$\text{spatial resolution} = 1.22\frac{\lambda}{D}f$$

3.3 Near field v.s. Far field diffraction

3.3.1 Derivation

The amplitude of wave at a point $P(x_0, y_0)$ on the diffraction plane is obtained by integrating over the aperture plane:

$$\psi_p = \int\int A\,h(x,y)\frac{e^{ikr}}{r}dxdy$$

$$\begin{aligned} r^2 &= L^2 + (x-x_0)^2 + (y-y_0)^2 \\ &= (L^2 + x_0^2 + y_0^2) - 2(x_0x + y_0y) + x^2 + y^2 \\ &= R^2\left(1 - \frac{2(x_0x + y_0y)}{R^2} + \frac{x^2+y^2}{R^2}\right) \\ r &= R\left(1 - \frac{2(x_0x + y_0y)}{R^2} + \frac{x^2+y^2}{R^2}\right)^{1/2} \end{aligned}$$

$$\Rightarrow \boxed{r \approx R \underbrace{- \frac{x_0x + y_0y}{R}}_{\text{Fraunhofer}} + \underbrace{\frac{x^2+y^2}{2R}}_{\text{Fresnel}}}$$

where R is the optical path length from the aperture plane to point P on the diffraction plane.

Rayleigh distance (between aperture and screen):

$$\boxed{d_R = \frac{a^2}{\lambda}}$$

where a is the maximum dimension of the aperture.

If distance $\gg d_R$, Fraunhofer regime (far-field). Discard quadratic term.
If distance $\ll d_R$, Fresnel regime (near-field). Preserve quadratic term but set it on-axis $x_0 = y_0 = 0$.

3.4 Fraunhofer Diffraction

$$\text{Diffraction pattern } = FT[\text{aperture function}]$$

$$\psi(p,q) \propto \int\int h(x,y)e^{-i(px+qy)}dxdy$$

where $p = k\sin\theta$ and $q = k\sin\phi$.

Condition for validity: when the phase variation across the aperture is linear in distance moved across the aperture when viewed from any point on the aperture screen of interest.

3.4.1 Babinet's principle

Babinet's principle states that: the intensity pattern due to aperture h is identical to that of its complementary aperture $h' = 1 - h(x, y)$, except at the origin.

Proof:

$$\begin{aligned}
\psi' &= A \int\int h'(x,y) e^{-i(px+qy)} dxdy \\
&= A \int\int (1-h) e^{-i(px+qy)} dxdy \\
&= A \int\int e^{-i(px+qy)} dxdy - \psi \\
&= \underbrace{4\pi^2 A \delta(p)\delta(q)}_{\text{vanishes off-axis}} - \psi
\end{aligned}$$

$$I' = |\psi'|^2 = |-\psi|^2 = |\psi|^2 = I$$

So, h and h' produces the same intensity pattern.

3.5 Key ideas

Convolution theorem to deconstruct an arbitrary aperture function to find the intensity pattern.

Position of maximums/peaks for narrow slits: $\boxed{q = n \times \frac{2\pi}{d}}$ or $\boxed{\sin\theta = \frac{n\lambda}{d}}$ where d is the slit spacing.

Position of first zero for top-hat function: $\boxed{q = \frac{2\pi}{a}}$ or $\boxed{\sin\theta = \frac{\lambda}{a}}$ where a is the width of the single slit.

Important Fourier transform parameter: $\boxed{q = k\sin\theta}$ where θ tells you the angle of diffraction.

3.6 Spectral line emission

1. **Natural line width**
Uncertainty principle tells us there is a small uncertainty in energy and hence the spectral line emitted has finite width. Fourier transform of the spectral line yields a Lorentzian power spectrum.

2. **Collision broadening**
Collisions between atoms while they are emitting limits the coherence of emitted light waves, giving a width of:

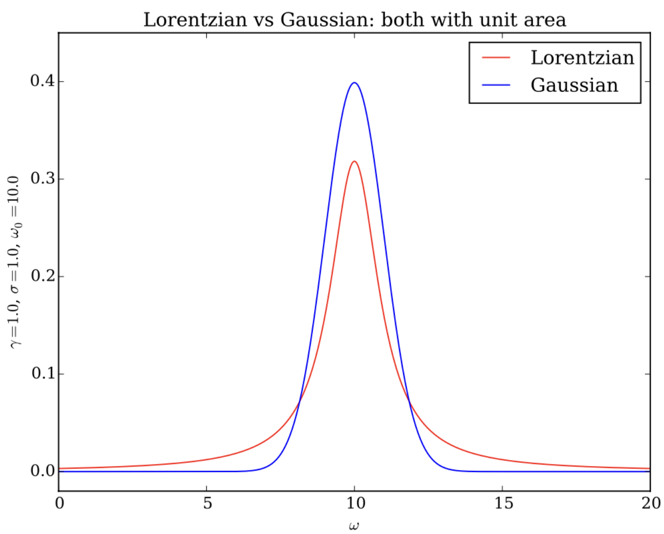
$$\boxed{\Delta\omega \sim n\sigma u}$$

where n is number density, σ is collision cross-section and u is particle speed.

3. **Thermal broadening**
When atoms in motion emit light, there will be Doppler shift, obtaining a Gaussian line with width:

$$\boxed{\sigma_\omega = \omega_0\sqrt{\frac{k_B T}{mc^2}}}$$

Lorentzian v.s. Gaussian



1. Lorentzian fall off slower.
2. In general, a profile is the convolution of Gaussian and Lorentzian.

3.7 Fresnel Diffraction

$$\psi_p \propto \int\int h(x,y)\, e^{ik\left(\frac{x^2+y^2}{2R}\right)} dxdy = \int_{x_1}^{x_2} e^{\frac{ikx^2}{2R}} dx \int_{y_1}^{y_2} e^{\frac{iky^2}{2R}} dy$$

$$= \underbrace{\int_{u_1}^{u_2} e^{\frac{i\pi u^2}{2}} du}_{C+iS \to \text{ Cornu spiral}} \int_{v_1}^{v_2} e^{\frac{i\pi v^2}{2}} dv$$

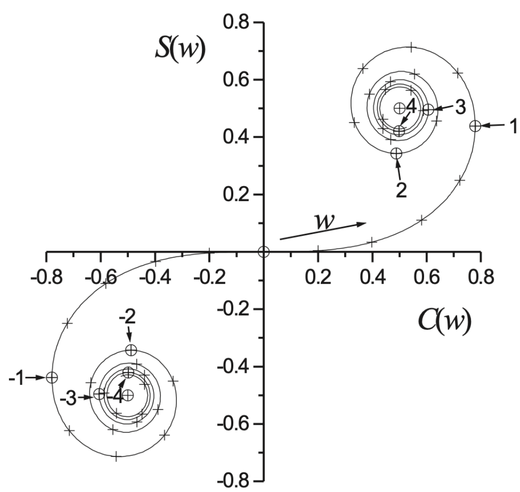
where

$$\boxed{u = x\sqrt{\frac{2}{\lambda R}} \quad \text{and} \quad v = y\sqrt{\frac{2}{\lambda R}}}$$

$$\boxed{R = \left(\frac{1}{a} + \frac{1}{b}\right)^{-1}}$$

where a is source-aperture distance and b is aperture-screen distance.

3.7.1 Cornu spiral

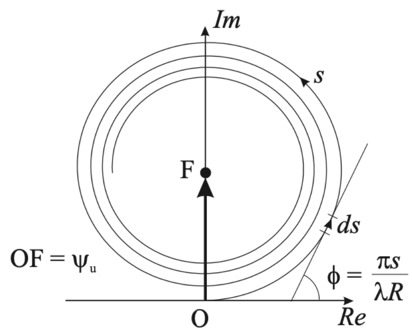


3.7.2 Circular aperture

For circular aperture of radius a, letting $s = x^2 + y^2 = r^2$ so $2r\,dr = ds$,

$$\psi_p \propto \int_0^a K e^{\frac{i\pi s}{\lambda R}} 2\pi r dr \propto \int_0^{a^2} K e^{\frac{i\pi s}{\lambda R}} ds$$

Phasor diagram:



- The integral can be thought of as the addition of lots of infinitesimal length of complex numbers (ds) with phase $\phi = \frac{\pi s}{\lambda R}$. As ds is very small, the sequence of phasors can be approximated as an arc of a circle.
- OF is the amplitude of undistorted wavefront. In the absence of obstruction, the integration range is from O to F.
- Radius of circle spirals inwards with increasing s due to $1/r$ dependence of amplitude and decreasing obliquity factor.
- At the origin, $\phi = 2n\pi$. At the top, $\phi = (2n+1)\pi$.

The first zone is up to the phase relative to the first being less than or equal to π:

$$\frac{\pi\rho^2}{\lambda R} \leq \pi \Rightarrow \rho^2 \leq \lambda R$$

The n^{th} zone satisfies:

$$(n-1)\pi \leq \frac{\pi\rho^2}{\lambda R} \leq n\pi$$

$$\boxed{\sqrt{(n-1)\lambda R} \leq \rho \leq \sqrt{n\lambda R}}$$

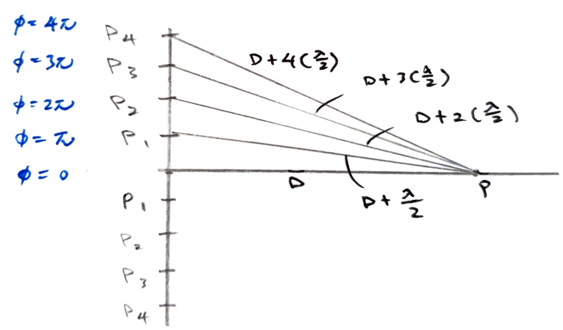


Figure 3: Another diagram interpretation

$$\boxed{\rho_n = \sqrt{n\lambda R}}$$

3.7.3 Poisson's spot

3.7.4 Fresnel zone plates

Blocks out alternate Fresnel half-period zones.

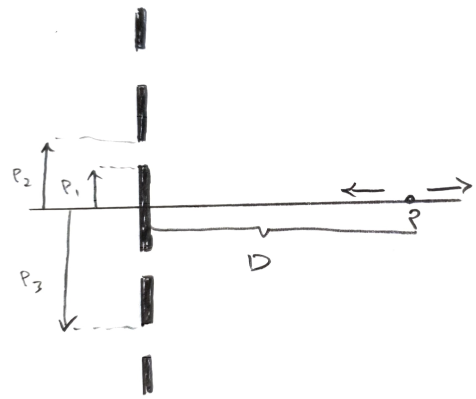
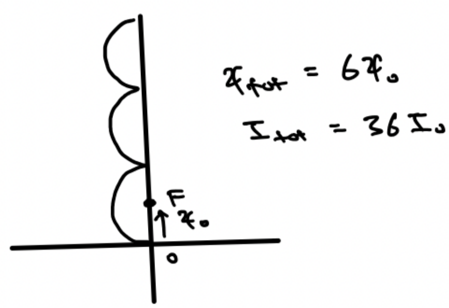


Figure 4: D is the focal length

For example, if the first three odd zones are blocked as shown in the above diagram i.e. only $n = 2, 4, 6$ get through, the phasor diagram consist of three half circles on the left side:



You get an intensity that is 36 times the unobstructed intensity.

Hence, we can design the zone plate to achieve **focusing** at particular distances.

Primary focal length:

$$\boxed{f_0 = \frac{\rho_1^2}{\lambda} = \frac{\rho_n^2}{n\lambda}}$$

4 Interference

4.1 Fourier Transform Spectroscopy

Consider superposition of 2 monochromatic waves,

$$\Psi = \Re[\psi_1 e^{-\omega_1 t} + \psi_2 e^{-\omega_2 t}]$$

Interference formula [using $\Re(A) = \frac{1}{2}(A + A^*)$]:

$$I(x) \propto \Psi^2 = \left(\frac{1}{2}\left[\psi_1 e^{-i\omega t} + \psi_2 e^{-i\omega t} + \psi_1^* e^{i\omega t} + \psi_2^* e^{i\omega t}\right]\right)^2$$

Considering $\langle I \rangle$, fast oscillating terms average to zero:

$$\boxed{\langle I \rangle = \frac{1}{2}\langle a_1^2 \rangle + \frac{1}{2}\langle a_2^2 \rangle + \langle a_1 a_2 \Re[e^{i(\Delta\phi - \Delta\omega t)}]\rangle}$$

where $\Delta\phi$ is the difference in phase and $\Delta\omega$ is the difference in angular frequency.

More generally for broadband light:

$$I(x) = I_1 + IFT[S(k)] = \boxed{I_1 + \int_{-\infty}^{\infty} S(k) e^{ikx}}$$

where $S(k)$ is the spectrum in k-space.

When doing Fourier transform, make the $S(k)$ symmetrical about the x-axis. Example: for a spectrum having a top-hat shape from $k - \Delta k/2$ to $k + \Delta k/2$, the $S(k)$ is a convolution of two delta function at k and $-k$ with a top-hat function of width Δk.

The reverse can be done to be find the spectrum of the source:

$$\boxed{S(k) \propto \int_{-\infty}^{\infty} [I(x) - I_1] e^{-ikx} dx}$$

$$\text{Fringe visibility } = \boxed{\frac{I_{max} - I_{min}}{I_{max} + I_{min}}}$$

Spectral resolution:

$$\boxed{\frac{\Delta\lambda}{\lambda} = \frac{\lambda}{2\Delta x}}$$

where Δx is the mirror displacement.

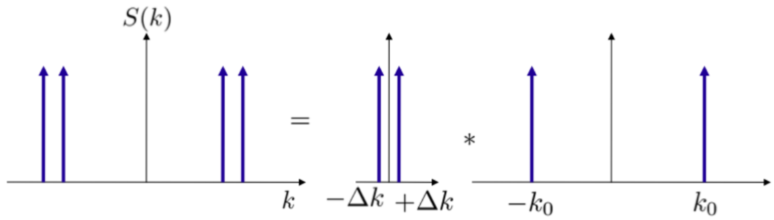
Examples:
(1) Dichromatic:

$$I(x) = I_0[1 + \cos k_0 x \cdot \cos \Delta k x]$$

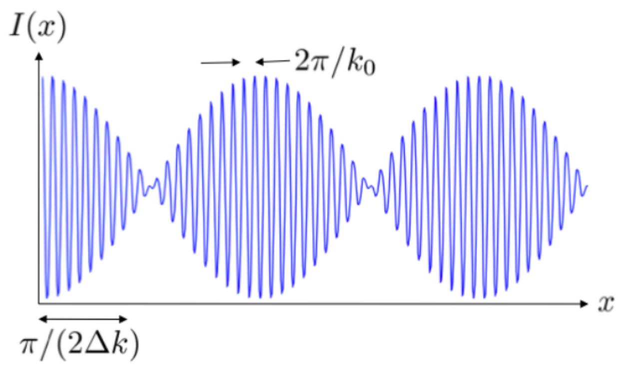
(2) Dichromatic with finite width spectral line:

$$I(x) = I_1[1 + e^{-\frac{(\Delta k x)^2}{2}} \cdot \cos k_0 x \cdot \cos \Delta k x]$$

Example: spectrum containing two closely-spaced wavelength components.
In k-space,



After Fourier transform, the intensity pattern is:



$$I(x) = I_0[1 + \cos k_0 x \cdot \cos \Delta k x]$$

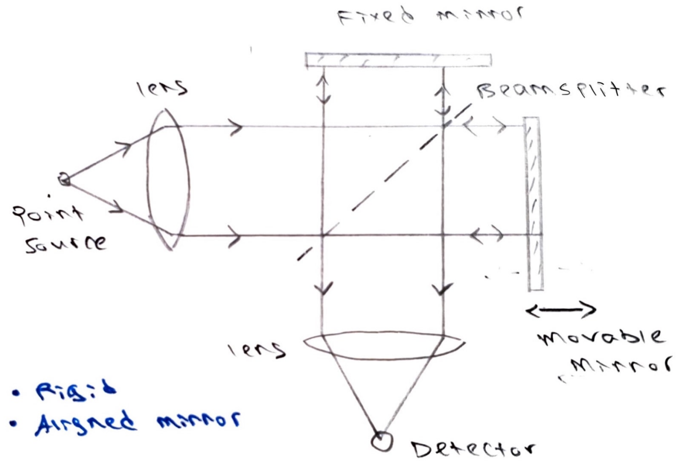
Envelope wavelength:

$$x_{env} = \frac{2\pi}{\Delta k}$$

Fast oscillation wavelength:

$$x_{fast} = \frac{2\pi}{k_0}$$

4.2 Michelson interferometer



1. How it works: light from a point source is transformed into parallel beams and then split at the beam splitter. They travel along different perpendicular paths and finally recombine and focused at the detector.

2. An interference pattern is observed at the detector. Constructive and destructive interference happens depending on the path difference of the two beams.

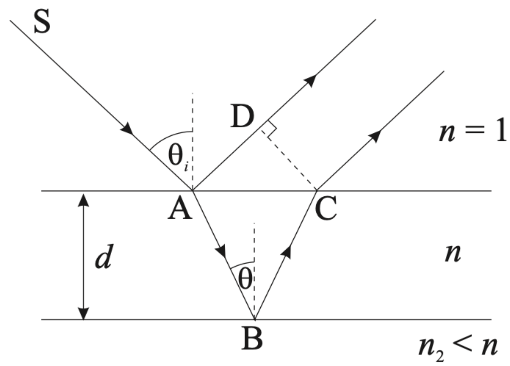
3. One can see fringe patterns by varying the displacement of the movable mirror to change path difference and plot the intensity at detector against mirror position.

Intensity pattern:
The two beams can be described by $ae^{i\omega t}$ and $ae^{i\omega t+i\phi}$. Note that $\phi = kx$,

Using interference formula:

$$I = 2 \times \frac{1}{2}\langle a^2 \rangle + \langle a^2 \Re[e^{ikx}] \rangle = \boxed{I_0(1 + \cos kx)}$$

4.3 Thin film interference



Interference between wave reflected at the top surface and the wave reflected from the bottom surface.

Note: if waves travelling in lower refractive index (higher impedance) medium reflect off higher refractive (lower impedance) index medium , phase change of π.

For near normal incidence, and where $\lambda' = \lambda/n$

(i) For minimum intensity, path difference $2d = m\lambda'$
(ii) For maximum intensity, path difference $2d = (m + \frac{1}{2})\lambda'$

Expression for **phase difference due to path difference** can be found by some trigonometry and Snell's law:

$$\boxed{\delta_{\text{path}} = 2\pi \times \frac{2 \cdot (d \cos\theta)}{\lambda'}}$$

where λ' is the wavelength in the medium.

Intensity pattern:

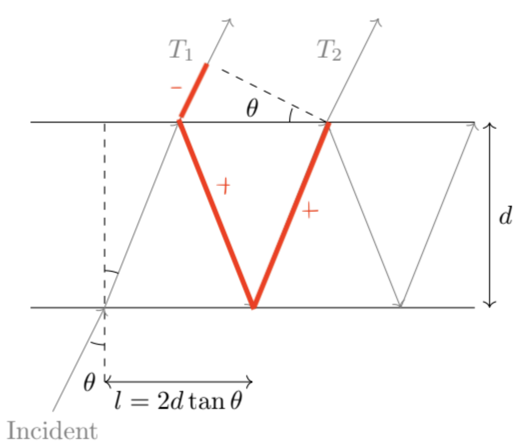
$$\boxed{I(\delta) = I_0 \left(1 - \Re[e^{i\delta}]\right)}$$

with the minus sign to account for the π phase change from reflection.

Examples: Newton's Ring (happens when air gap is formed between surfaces, used in quality control of optical surfaces), soap films, thin film of oil on water.

4.4 Fabry-Perot Etalon (multi-beam interference)

High resolution spectroscopy.



$$\begin{aligned}\text{o.p.d.} &= 2\sec\theta - l\sin\theta \\ &= 2d\left(\frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta}\right) \\ &= \boxed{2d\cos\theta}\end{aligned}$$

$$\text{extra phase for round trip: } \delta = k \times (2d\cos\theta)$$

$$\text{Amplitude transmission coeff: } A = T\underbrace{\left(1 + Re^{i\delta} + R^2e^{2i\delta} + R^3e^{3i\delta} + ...\right)}_{\text{Geometric progression}} = \boxed{\frac{T}{1 - Re^{i\delta}}}$$

$$\text{Intensity transmission coeff: } B = AA^* = \frac{T^2}{1 + R^2 - 2R\cos\delta}$$

The peaks are located at $\cos\delta = 1$ so $\delta = 2m\pi$, (assuming normal incidence)

Finding position of half-width at half maximum:

$$\text{Half power points: } \frac{T^2}{2(1-R)^2} = \underbrace{\frac{T^2}{(1-R)^2}}_{\text{maximum intensity}} \left(\frac{1}{1 + \frac{4R}{(1-R)^2}\sin^2(\delta_{1/2}/2)}\right)$$

Solve for $\delta_{1/2}$.

Free spectral range is the wavelength difference at which overlapping takes place.

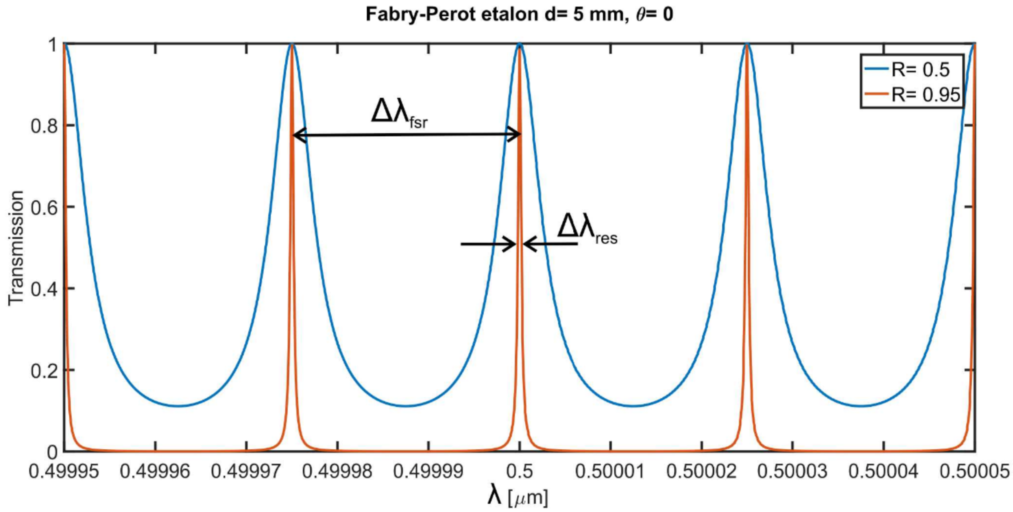
$$-\frac{4\pi d}{\lambda} = 2m\pi \Rightarrow \frac{2d}{\lambda} = m$$

$$\therefore -\frac{2d}{\lambda^2}\Delta\lambda = \Delta m = 1 \quad \text{(for neighbouring peaks)}$$

$$d_{crit} = \frac{\lambda^2}{2\Delta\lambda} \quad \text{and} \quad \Delta\lambda = \frac{\lambda}{m}$$

Benefits: very fine resolution of closely spaced peaks.
Drawback: suffer from free spectral range problem, as $\Delta\lambda$ increases, d_{crit} shrinks to impossible values.

The intensity graph looks like this:


Fabry-Perot etalon d= 5 mm, θ= 0
R= 0.5
R= 0.95
Δλfsr
Δλres
Transmission
λ [μm]
0
0.2
0.4
0.6
0.8
1
0.49995
0.49996
0.49997
0.49998
0.49999
0.5
0.50001
0.50002
0.50003
0.50004
0.50005