Physics B - Classical Mechanics Summary Notes

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Write down the Physics, then just do the Math :)

1 Fundamentals of classical dynamics

Momentum:

$$\vec{p} = m\dot{\vec{r}}$$

Impulse:

$$\int \vec{F} dt = \Delta \vec{P}$$

Force:

$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}\vec{v}$$

Conservation law: conservation of momentum in the absence of external forces

Rotational equivalent

Angular momentum:

$$\vec{J} = \vec{r} \times \vec{p} = I\vec{\omega}$$

Impulse:

$$\int \vec{\tau}\, dt = \Delta \vec{J}$$

Torque:

$$\vec{\tau} = \vec{r} \times \vec{F} = I\vec{\alpha}$$

Conservation law: conservation of angular momentum in the absence of external torque

Energy

Work done:

$$dW = \vec{F} \cdot d\vec{r}$$

Kinetic energy:

$$T = \frac{1}{2}m\dot{x}^2$$

Potential energy:

$$V = \frac{1}{2}kx^2$$

Conservation law: conservation of energy in the absence of dissipative forces

1.1 Techniques for solving for equation of motion

1. Force method
2. Energy method
$$E = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$
$$\frac{dE}{dt} = 0 \ \text{(COE)}$$
3. Lagrangian method
$$\mathcal{L} = T - V$$
$$\frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{q}_i}\right) = \frac{\delta \mathcal{L}}{\delta q_i} \ \text{(Euler-Lagrange)}$$

1.2 Many particle systems

COM:
$$M\vec{R}_{COM} = \sum_a m_a \vec{r}_a$$

Total external force → motion of COM
Total external torque → net change in $\vec{J}$
Work done by external force → change in energy dE

1.3 Zero-momentum frame / COM frame

A frame where $\vec{P}' = 0$ i.e. COM is stationary
$$\vec{r}\,' = \vec{r} - \vec{v}t$$
where $\vec{v} = \frac{\vec{P}}{M}$ which is the velocity of COM

1.4 Coordinate system

Plane polar coordinates: $\dot{\underset{\sim}{e}}_\rho = \dot{\phi}\, \hat{\underset{\sim}{e}}_\phi$ and $\dot{\underset{\sim}{e}}_\phi = -\dot{\phi}\, \hat{\underset{\sim}{e}}_\rho$

Useful representation in complex form: $\hat{\underset{\sim}{e}}_\rho = e^{i\phi}$ and $\hat{\underset{\sim}{e}}_\phi = ie^{i\phi}$

$$\underset{\sim}{r} = \rho\, \hat{\underset{\sim}{e}}_\rho$$
$$\dot{\underset{\sim}{r}} = \dot{\rho}\, \hat{\underset{\sim}{e}}_\rho + \rho\dot{\phi}\, \hat{\underset{\sim}{e}}_\phi$$
$$\ddot{\underset{\sim}{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\, \hat{\underset{\sim}{e}}_\rho + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\, \hat{\underset{\sim}{e}}_\phi = \underbrace{(\ddot{\rho} - \rho\dot{\phi}^2)}_{\text{radial}}\, \hat{\underset{\sim}{e}}_\rho + \underbrace{\frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\phi})}_{\text{tangential}}\, \hat{\underset{\sim}{e}}_\phi$$

1.5 Rotating frames

Fictitious forces appear in non-inertial frames such as accelerated frames or rotating frames. Examples are coriolis force or centrifugal force.

Frame S rotates with angular velocity $\vec{\omega}$ w.r.t frame S_0.

1. Using $\dot{\underset{\sim}{\hat{e}}}_i = \vec{\omega} \times \underset{\sim}{\hat{e}}_i$ (from $\vec{v} = \vec{\omega} \times \vec{r}$) where $i = \{x, y, z\}$

$$\underset{\sim}{r_0} = x\,\underset{\sim}{\hat{e}_x} + y\,\underset{\sim}{\hat{e}_y} + z\,\underset{\sim}{\hat{e}_z}$$

$$\dot{\underset{\sim}{r_0}} = \dot{x}\,\underset{\sim}{\hat{e}_x} + x\,\dot{\underset{\sim}{\hat{e}_x}} + ... = \underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r}$$

$$\ddot{\underset{\sim}{r_0}} = \ddot{x}\,\underset{\sim}{\hat{e}_x} + 2\dot{x}\,\dot{\underset{\sim}{\hat{e}_x}} + x\,\ddot{\underset{\sim}{\hat{e}_x}} ... = \boxed{\underset{\sim}{a} + 2(\underset{\sim}{\omega} \times \underset{\sim}{v}) + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})}$$

$$\boxed{\underbrace{m\underset{\sim}{a}}_{\text{apparent}} = \underbrace{m\ddot{\underset{\sim}{r_0}}}_{\text{real}} \underbrace{- \underbrace{2m(\underset{\sim}{\omega} \times \underset{\sim}{v})}_{\text{coriolis force}} - \underbrace{m\underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})}_{\text{centrifugal force}} - m\ddot{R}}_{\text{fictitious force}}}$$

Coriolis force: appear if body if moving with respect to a rotating frame

- Related to conservation of angular momentum
- Coriolis force:

$$\boxed{\boldsymbol{a_{cor}} = -2\,\boldsymbol{\Omega} \times \mathbf{v}}$$

For motion along surface, force is always to the right in the Northern hemisphere, and always to the left in the Southern hemisphere. Just do the cross product.

Responsible for weather patterns on Earth as well as can be observed from the Foucault pendulum, which precesses at $\Omega \sin\lambda$ due to the Coriolis force.

Centrifugal force: give rise to Earth's equatorial bulge.

2. Using operator:

$$\boxed{\left[\frac{d}{dt}\right]_{S_0} = \left[\frac{d}{dt}\right]_S + \underset{\sim}{\omega}\times}$$

$$\text{Expand } \left[\frac{d^2\underset{\sim}{r}}{dt^2}\right]_{S_0} = \left(\left[\frac{d}{dt}\right]_S + \underset{\sim}{\omega}\times\right)\left(\left[\frac{d\underset{\sim}{r}}{dt}\right]_S + \underset{\sim}{\omega} \times \underset{\sim}{r}\right)$$

1.6 Lagrangian Mechanics

Starts with Hamilton's Principle that the action $S = \int \mathcal{L}(q_i, \dot{q}_i, t)dt$ is stationary for small variations $\delta q_i(t)$ about $q_i(t)$.

$$\delta S = \int_{t_1}^{t_2} \sum_i \left(\delta q_i \frac{\partial \mathcal{L}}{\partial q_i} + \delta \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) dt$$

Integrating by parts:

$$\delta S = \underbrace{\ldots}_{\substack{\text{vanish for} \\ \text{fixed end} \\ \text{points}}} + \int_{t_1}^{t_2} \sum_i \delta q_i \left[\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \right] dt$$

For $\delta S = 0 \ \forall \ \delta q_i$, we get the **Euler-Lagrange equation**:

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)}$$

1.6.1 Conjugate momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

If the Lagrangian is independent of coordinates q_1, then the conjugate momentum p_1 is conserved (a constant).

Symmetries lead to conservation laws (Noether's Theorem)

1.6.2 Examples

1. Simple harmonic motion
$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$
2. Orbits in central potential
$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$
3. Symmetric top
$$\mathcal{L} = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 - mgh\cos\theta$$
4. Ladder problem
5. Double pendulum

1.7 Hamiltonian Dynamics

Find $H(q_i, p_i, t)$ that does not depend on velocities.

Defining:

$$\boxed{H = \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t)}$$

$$dH = \sum_i \left(\dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \right) - \frac{\partial \mathcal{L}}{\partial t} dt$$

Since $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$ and $\frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i$m

$$dH = \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial \mathcal{L}}{\partial t} dt$$

H only responds to changes in q_i, p_i and t, as we wanted.

Comparing with,

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt$$

We can get expressions for $\dot{q}_i$ and $\dot{p}_i$.

$$\boxed{\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}}$$

If the Lagrangian does not depend on time, the Hamiltonian is conserved.

1.8 Tripos Q & A

Q1. (2015 Tripos P1 A2) Two people stand diametrically opposite each other on the edges of a horizontal circular platform of radius a rotating about its axis at angular speed ω. One throws a ball directly at the other at speed $v \gg a\omega$. By what horizontal distance will the ball miss the second person?

A1: There is coriolis force.

$$a_c = -2\omega v$$

$$x = \frac{1}{2}at^2 = \frac{1}{2}(2\omega v)\left(\frac{2a}{v}\right)^2 = \frac{4\omega a^2}{v}$$

2 Orbits in central force field

$$\text{Central field: } \underset{\sim}{F} = -\nabla U = -\frac{dU}{dr}\underset{\sim}{\hat{e}_r}$$

Obeys conservation laws/most important equations:

1. Angular momentum conserved as $\mathbf{G} = \mathbf{r} \times \mathbf{F} = 0$ or Lagrangian is independent of ϕ:

$$\boxed{J = mr^2\dot{\phi} = constant}$$

2. Total energy conserved

$$\begin{aligned} E &= U(r) + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \\ &= \boxed{\frac{1}{2}m\dot{r}^2 + \underbrace{U(r) + \frac{J^2}{2mr^2}}_{\text{effective potential}}} \end{aligned}$$

Effective potential: by using a constant of the motion, we have removed the ϕ dependence i.e. reduced the degree of freedom in the system by one.

For a general power-law force field: $F = -Ar^n \Rightarrow U(r) = \frac{Ar^{n+1}}{n+1}$

2.1 Classifying orbits

Define r_0 to be where $\frac{dU_{\text{eff}}}{dr}|_{r_0} = 0$

1. stable v.s. unstable

$\frac{d^2U_{\text{eff}}}{dr^2}\Big|_{r_0} > 0$ is stable

2. bound v.s. unbound

$E < 0$ is bound. $E > 0$ is unbound

2.2 Nearly-circular orbits (perturbations about r_0)

Two ways to find equation of motion: $m\ddot{\epsilon} + U''_{\text{eff}}\epsilon = 0$:

1. Expand U_{eff} about r_0

$$U_{\text{eff}} = U_0 + \frac{1}{2}(r - r_0)^2 \left.\frac{d^2U_{\text{eff}}}{dr^2}\right|_{r_0}$$

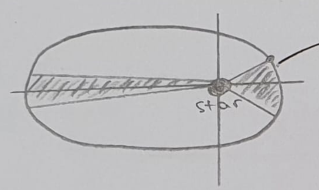
Spot that $\frac{d^2U_{\text{eff}}}{dr^2}$ is a "spring constant".

2. Energy method

$$\frac{d(U_{\text{eff}} + \frac{1}{2}m\dot{r}^2)}{dt} = 0$$

3. Expand equation of motion about r_0 i.e. $r \to r_0 + \delta r$

2.3 Kepler's Laws



Kepler's 2^{nd} law

↳ equal area sweeped in equal times

Kepler's 3^{rd} law

↳ $T^2 = \frac{4\pi^2}{GM} a^3$

Kepler's 1^{st} law

↳ elliptical orbit with star at one focus

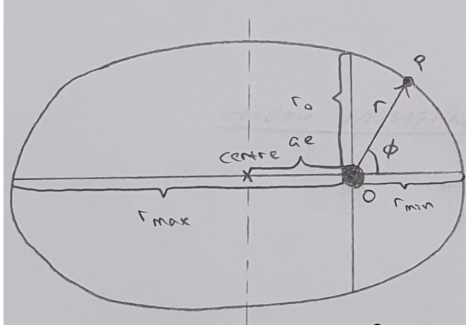
Fast derivation of Kepler's third law:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \quad \text{and} \quad v = \frac{2\pi r}{T}$$

$$\frac{GM}{r} = v^2 = \frac{4\pi^2 r^2}{T^2}$$

$$\Rightarrow \boxed{T^2 = \frac{4\pi^2}{GM} r^3}$$

2.4 Geometry of an ellipse



Polar equation: $r_0 = r(1 + e\cos\phi)$.

$x = r\cos\phi \Rightarrow r = r_0 - ex$

$x^2 + y^2 = r^2 = (r_0 - ex)^2$

$\frac{x'^2}{a^2} + \frac{y^2}{b^2} = 1$, $x' = x + \frac{r_0 e}{1-e^2}$

$r_{min} = \frac{r_0}{1+e}$ $\quad a = \frac{r_0}{1-e^2}$ $\quad r_{max} = a(1+e)$

$r_{max} = \frac{r_0}{1-e}$ $\quad b = \frac{r_0}{\sqrt{1-e^2}}$ $\quad r_{min} = a(1-e)$

$r_{max} + r_{min} = 2a$.

Area of ellipse: $\pi ab = \frac{\pi r_0^2}{(1-e^2)^{3/2}}$, rate of sweeping unit area $= \frac{1}{2} r^2 \dot{\phi}$ } can find period T.

2.5 Interested in: inverse square law force $(n = -2)$

$$F = -\frac{A}{r^2}, \;\; U = -\frac{A}{r}$$

Gravity: $A = GMm$
Electric: $A = -\frac{q_1 q_2}{4\pi\epsilon_0}$

We can write down J and E and that is enough to know everything about the system.

Problem solving tip: **conserve energy and angular momentum**.

<u>Overview of orbit shapes and conditions</u>:

For inverse-square force field for an **attractive** force,

- All orbits are conic sections with the centre of force at the focus.
- Equation for shape of orbit is of the form:

$$\boxed{r = \frac{r_0}{1 + e\cos\theta}}$$

- For $E > 0$: orbit is hyperbolic $(e > 1)$
  $E = 0$: orbit is parabolic $(e = 1)$
  $E < 0$: orbit is either circular $(e = 0)$ or elliptical $(0 < e < 1)$

For **repulsive** field: only hyperbolic orbits are possible.

2.5.1 Derivation of shape and energy of orbit

Write down energy and angular momentum:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V \quad \text{and} \quad J = mr^2\dot{\phi}$$

Divide by $\boldsymbol{\dot{\phi}^2 = \frac{J^2}{m^2 r^4}}$ and multiply by $\frac{2}{m}$ to make coefficient of $\left(\frac{dr}{d\phi}\right)^2$ unity. Obtain:

$$\frac{2mE}{J^2}r^4 = \left(\frac{dr}{d\phi}\right)^2 + r^2 + V(r)\frac{2mr^4}{J^2}$$

Make substitution $u = 1/r$. Do the algebra.

Alternatively, using a vectors approach, note that $\underset{\sim}{J}$, $\dot{\underset{\sim}{v}}$ and $\dot{\hat{\underset{\sim}{e}}}_r$ are mutually perpendicular.

$$J = mr^2\dot{\phi} \quad \text{and} \quad \dot{v} = \frac{1}{m}\frac{A}{r^2} \quad \text{and} \quad |\dot{\hat{\underset{\sim}{e}}}_r| = \dot{\phi}$$

<u>Shape</u>

$$\underset{\sim}{J} \times \dot{\underset{\sim}{v}} = -A\dot{\hat{\underset{\sim}{e}}}_r \quad \text{(by inspection)}$$

$$\downarrow \quad \text{integrate (1), dot with } \underset{\sim}{r}$$

$$\boxed{\text{Attractive} : r_0 = r(1 + e\cos(\phi)) = \frac{J^2}{mA}}$$

$$\boxed{J^2 = Amr_0}$$

$$\text{Repulsive} : r_0 = r(e\cos(\phi) - 1) \quad \text{[other branch]}$$

<u>Energy</u>

$$A\underset{\sim}{e} = -(\underset{\sim}{J} \times \underset{\sim}{v} + A\dot{\hat{\underset{\sim}{e}}}_r) \quad \text{result of (1)}$$

$$\downarrow \quad \text{scalar product with itself, rearrange}$$

$$\boxed{e^2 - 1 = \frac{2r_0E}{A}}$$

$$\boxed{E = -\frac{A}{2a}}$$

2.6 Parabolic and hyperbolic orbits

2.6.1 Parabolic

Parameters: $e = 1, E = 0$

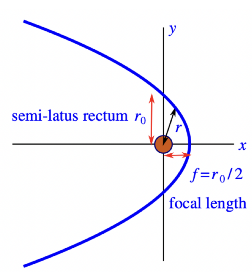
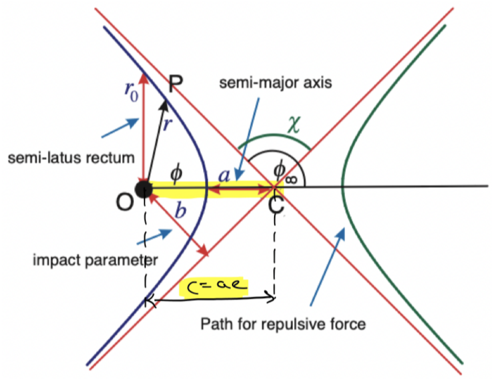
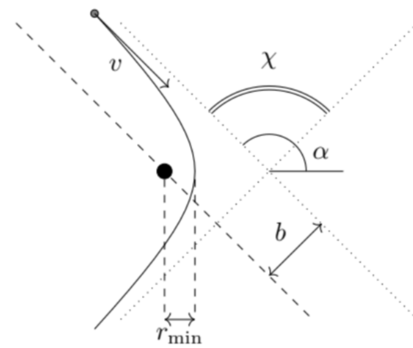


Figure 1: $y^2 = 4f(f - x)$

2.6.2 Hyperbolic

Parameters: $e > 1, E > 0$





where χ is angle of deflection and $\phi_\infty = \alpha$ is the angle of approach from the horizontal.

$$\text{At infinity,} \quad J = mbv_\infty \quad \text{and} \quad E = \frac{1}{2}mv_\infty^2$$

Key ideas:

1. Angle relationships:

$$\boxed{2(\alpha - \chi) + \chi = \pi}$$

$$\Rightarrow \alpha = \frac{1}{2}(\pi + \chi)$$

2. Conserve energy and angular momentum to find relevant parameters.

3. Consider change in momentum to find the following formula:

$$\boxed{\tan\left(\frac{\chi}{2}\right) = \frac{A}{mbv_\infty^2}}$$

Derivation: force acting over a certain time cause change in momentum.

At infinity initially, there is horizontal component to the angular momentum, but at the closest approach, all the horizontal momentum is gone. It is due to horizontal forces acting. We can write:

$$mv_\infty \cos\alpha = \int_{\text{initial}}^{\text{closest}} \frac{A}{r^2} \cos\theta dt$$

Trick: $dt = d\theta/\dot{\theta}$ and $\dot{\theta} = \frac{J}{mr^2}$. Putting together we have $dt = \frac{mr^2}{J}d\theta$

$$mv_\infty \cos\alpha = \frac{Am}{J}\int_\alpha^0 \cos\theta d\theta = \frac{A}{bv_\infty}(-\sin\alpha)$$

Finally,

$$\boxed{\frac{A}{mbv_\infty^2} = -\frac{\cos\alpha}{\sin\alpha}}$$

4.
$$\boxed{\cos(\alpha) = -\frac{1}{e}}$$
$$\Rightarrow \tan^2(\phi_\infty) = e^2 - 1 \quad \text{(gradient of asymptote)}$$

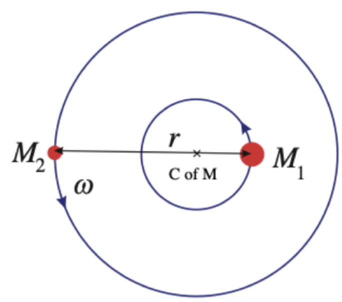
2.7 Useful equations

Vis-viva equation:
$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

Virial theorem:
$$\langle T \rangle = -\langle E \rangle = -\frac{1}{2}\langle U \rangle$$

2.8 Two-body problem

Two masses will orbit about their centre of mass.



$$r_1 = \frac{M_2}{M_1 + M_2} r$$

Defining:

1. $\underset{\sim}{r} = \underset{\sim}{r_1} - \underset{\sim}{r_2}$
2. $\mu = \frac{M_1 M_2}{M_1 + M_2}$ (reduced mass)

All two-body problems can be reduced to a **one-body problem** in the centre of mass frame using **reduced mass**.

We have:
$$\mu r \omega^2 = \frac{GM_1M_2}{r^2} \quad \text{(force balance)}$$
$$T = \frac{1}{2}\mu \dot{r}^2 \quad \text{(kinetic energy)}$$
$$\underset{\sim}{J} = \mu \underset{\sim}{r} \times \dot{\underset{\sim}{r}} \quad \text{(angular momentum)}$$

2.9 Gravity

1. Gravitational potential and field - determines energies and accelerations

$$\phi = -\frac{GM}{R} \quad \text{and} \quad |g| = \frac{GM}{R^2}$$

2. Gauss's Law:

$$\int_V g \cdot d\mathbf{A} = -4\pi G M_{\text{encl.}}$$

3. Gravitational tidal field - due to non-uniformity of gravitational field causing a finite sized object to feel different forces at different points.

$$\underset{\sim}{T}(\vec{a}) = \vec{a} \cdot \nabla \underset{\sim}{g} = a_j \frac{\delta g_i}{\delta x_j} \quad \text{(difference in } \underset{\sim}{g} \text{ between 2 locations)}$$

where $\vec{a}$ is the change in position. It is a **second-order effect**.

Derivation of radial tidal acceleration
Due to <u>variation of g in the radial direction</u>, causing a net stretch.

Expand:

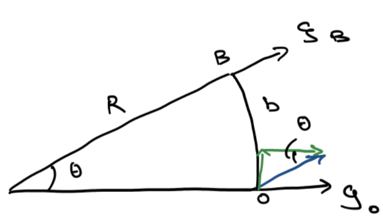
$$\underset{\sim}{g}_A = -\frac{GM}{(R+a)^2} \approx -\frac{GM}{R^2}\left(1 - 2\frac{a}{R} \; ...\right)$$

So the tidal acceleration, T for unit change in radial position is:

$$\boxed{T_r = \frac{2GM}{R^3}}$$

Derivation of sideways tidal acceleration
Due to <u>different direction of g at the two ends of the body</u>, causing a net compression.

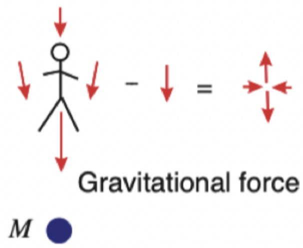
Consider a small offset b in the $\underset{\sim}{\hat{e}}_\theta$ direction, with $\theta = \frac{b}{R}$.



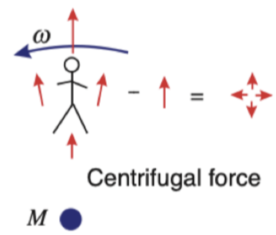
$$\Delta \underset{\sim}{g}_\theta = -\sin\theta \frac{GM}{R^2} \approx -\frac{GM}{R^2}\frac{b}{R}$$

So the tidal acceleration, T for unit change in position in the θ direction is:

$$\boxed{T_\theta = -\frac{GM}{R^3}}$$



(a) Radial stretching and sideways compression



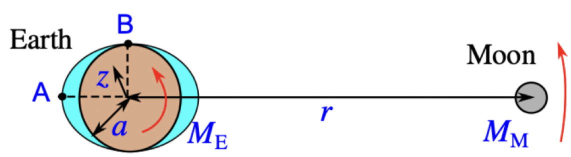
(b) Stick man is co-rotating with the orbit, there is contribution from centrifugal force

Co-rotating
Introduces additional centrifugal forces.

$$f_c = ma\omega^2$$

where a is the radius of the body. To keep the same orientation, the ω^2 must equal to the orbital angular frequency.

Application to Earth-Moon system



$$\phi_{tidal} = \int_0^a \underset{\sim}{g}_{T,A} - \underset{\sim}{g}_{T,B}\, dz$$

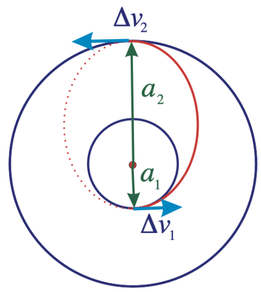
Can be used to calculate height of tides.

Frequency of precession:

$$\omega_p = \Omega - \omega$$

where Ω is the angular speed for circular orbit and ω is the angular frequency of small perturbation to the orbit.

Hohmann transfer orbit



- Hohmann transfer orbit is half of the elliptical orbit that touches both the initial circular orbit and the desired circular orbit. The elliptical orbit has $2a = a_1 + a_2$

- Using Virial theorem ($E = \frac{1}{2}U$). Initial energy $E_1 = -\frac{GMm}{2a_1}$ and $v_1 = \sqrt{\frac{GM}{a_1}}$. Has to be increased to:

$$E_{\text{transfer}} = -\frac{GMm}{a_1 + a_2}$$

  At point 1, boost velocity by Δv_1 to enter elliptical transfer orbit. At point 2, boost velocity by Δv_2 to enter the bigger circular orbit.

- Energy after boosting:

$$E_{\text{transfer}} = -\frac{GMm}{a_1} + \frac{1}{2}mv_{t1}^2 = -\frac{GMm}{a_1 + a_2}$$

- We can find v_{t1}

$$\Delta v = v_{t1} - v_1 = \sqrt{\frac{2GMa_2}{a_1(a_2 + a_1)}} - \sqrt{\frac{GM}{a_1}}$$

- Hohmann transfer is the most fuel efficient orbit in the absence of large bodies nearby.

Radial impulse do not change angular momentum and do not change energy. They simply change the SHAPE of the orbit E.g. change from elliptical to circular orbit.

Approach to find radius of new circular orbit:
1. Write down conservation of energy and angular momentum at the perihelion and aphelion.
2. Write down condition for circular orbit in terms of L.
3. Solve for r, radius of circular orbit.

3 Rigid Body Dynamics

Angular momentum

$$\underset{\sim}{J} = \sum \underset{\sim}{r} \times \underset{\sim}{p} = \sum \underset{\sim}{r} \times (\underset{\sim}{r} \times \underset{\sim}{\omega}) = \underbrace{\sum m r^2 \omega}_{\text{diagonal terms}} - \underbrace{\sum m \underset{\sim}{r}(\underset{\sim}{\omega} \cdot \underset{\sim}{r})}_{\text{off-diagonal terms}} = \underline{\underline{I}} \cdot \underset{\sim}{\omega}$$

$$\Rightarrow \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \underbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}}_{\text{symmetrical}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

with

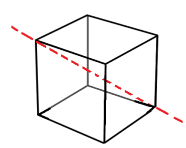
$$I_{x_1 x_1} = \int_V x_2^2 + x_3^2 \; \rho dV$$

$$I_{x_1 x_2} = -\int_V x_1 x_2 \rho dV$$

Moment of inertia tensor can be **diagonalised** to give the principal moment of inertia $\{I_1, I_2, I_3\}$ (eigenvalues) and the principal axes $\{\underset{\sim}{\hat{e}_1}, \underset{\sim}{\hat{e}_2}, \underset{\sim}{\hat{e}_3}\}$(eigenvectors).
Trivially,

$$\underset{\sim}{J} = \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix}$$

Example: Moment of inertia of a cube with length 2a (2012 Tripos P1 A4)



Taking moment of inertia about centre of cube:

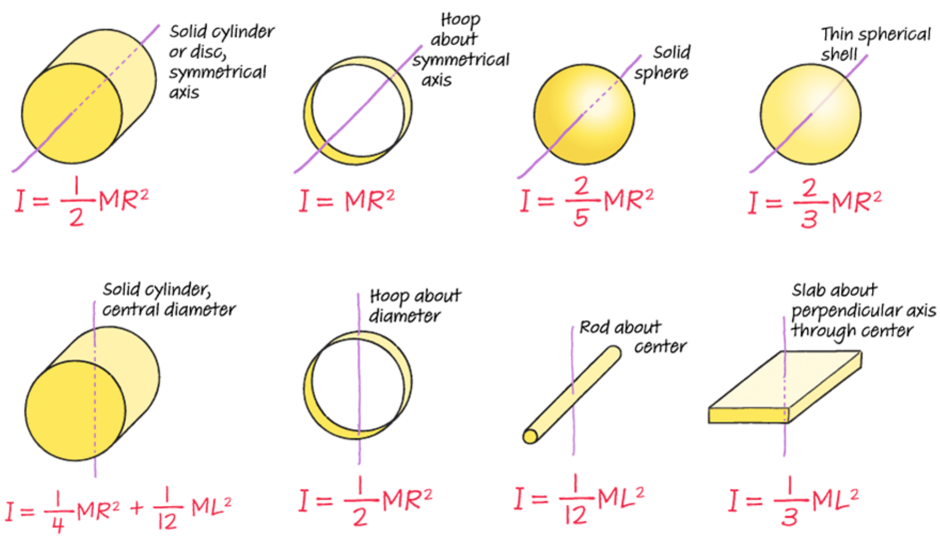
$$I_{yy} = I_{zz} = I_{xx} = \underbrace{\frac{M}{(2a)^3}}_{\text{density}} \int_{-a}^{a} \int_{-a}^{a} \int_{-a}^{a} (y^2 + z^2) \; dx dy dz = \frac{2}{3} M a^2$$

$$I_{xy} = -\frac{M}{(2a)^3} \int_{-a}^{a} \int_{-a}^{a} \int_{-a}^{a} xy \; dx dy dz = 0$$

$$I_{cube} = \frac{2}{3} M a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Moment of inertia tensor is rotation invariant/isotropic, the moment of inertia tensor taken along the body diagonals is equal to I_{cube}.

Common values of moment of inertia



Kinetic energy

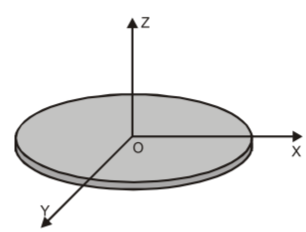
$$T = \frac{1}{2}\underset{\sim}{\omega} \cdot \underset{\sim}{J} = \frac{1}{2}\,\omega_i I_{ij} \omega_j$$

$$\boxed{T = \frac{1}{2}\,\underline{J}\;\underline{\underline{I}}^{-1}\;\underline{J}}$$

Note: $\vec{v} = \omega \times \vec{r}$

Useful theorems

1. Perpendicular axis theorem. For a lamina (flat body):



$$I_x + I_y = I_z$$

2. Parallel axis theorem: used to relate moment of inertia off-centre (I') to moment of inertia through centre (I).

$$I'_{ij} = I_{ij} + M(\underbrace{\delta_{ij} a^2}_{\text{diagonal terms}} - \underbrace{a_i a_j}_{\text{off-diagonal terms}})$$

3.1 Problem solving

Always think about energy and momentum. The physics is just:

1. Conservation of energy
2. Conservation of momentum / considering impulse (both linear and angular)
   For angular impulse: chosen to be about a particular axis of which its moment of inertia has to be calculated

3.2 General precession

Note for free precession, $F = 0$ and $G = 0$.

Equation of motion:

$$\underset{\sim}{G} = \left[\frac{d\underset{\sim}{J}}{dt}\right]_S + \underset{\sim}{\omega} \times \underset{\sim}{J} \quad \text{(rotation operator)}$$

$$\Rightarrow \boxed{\begin{cases} G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 \\ G_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \\ G_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 \end{cases}} \quad \text{(Euler's equation)}$$

(Seriously, Euler's equation is just equation of motion in a rotating frame uh, nothing fancy)

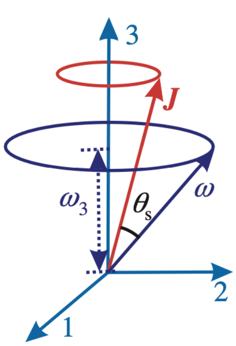
Geometrically, precession can be represented by rolling of space cone and body cone (Poinsot's)

3.2.1 Body and space frame

For symmetric body with $I_1 = I_2 \neq I_3$,

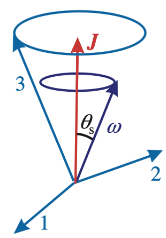
In the body frame, $\underset{\sim}{\omega}$ precesses about the unique 3-axis at the body frequency, sweeping out the body cone (θ_b):

$$\boxed{\Omega_b = \frac{I_1 - I_3}{I_1}\omega_3 = \left(1 - \frac{I_3}{I_1}\right)\omega_3}$$



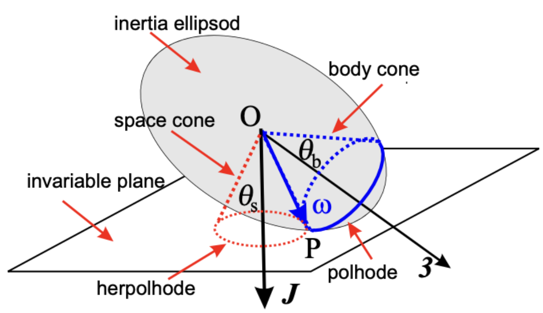
In the space frame (inertial observer frame), $\underset{\sim}{\omega}$ precesses about $\underset{\sim}{J}$ at the space frequency, sweeping out the space cone (θ_s):

$$\boxed{\Omega_s = \frac{|J|}{I_1}}$$



3.2.2 Body and space cone (Poinsot's treatment)

Key idea: can calculate space and body frequency from geometric considerations.



1. For free precession, constant $\mathbf{J}$ and $T(=\frac{1}{2}\omega \cdot \mathbf{J})$ implies component of ω in the direction of $\mathbf{J}$ remains constant.
2. As the ω vector varies, it keeps its tip in a plane perpendicular to $\mathbf{J}$, called the invariable plane.
3. The ellipsoid is tangential to the invariable plane at P.
4. Since the instantaneous motion is a rotation about OP, P is instantaneously at rest so the ellipsoid rolls rather than slides on the invariable plane.

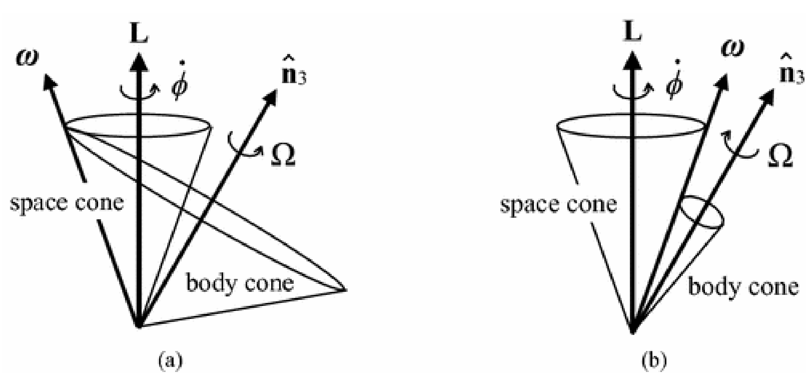
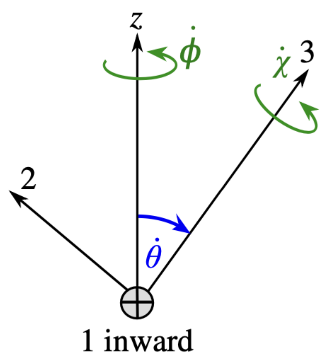


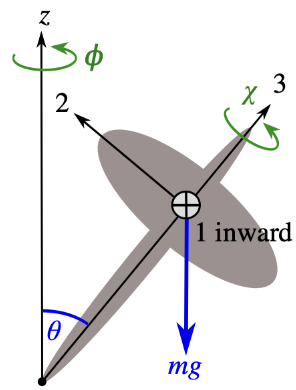
Figure 4: (a) prolate (b) oblate

Key argument: the **body cone rolls without slipping around the space cone**.

$$\Omega_b \sin\theta_b = \Omega_s \sin\theta_s$$

3.2.3 Coordinates system (Lagrange's treatment)





$$\underset{\sim}{\omega} = \dot{\phi}\,\underset{\sim}{\hat{e}_z} + \dot{\theta}\,\underset{\sim}{\hat{e}_1} + \dot{\chi}\,\underset{\sim}{\hat{e}_3}$$

Useful relation:

$$\boxed{\underset{\sim}{\hat{e}_z} = \underset{\sim}{\hat{e}_3}\cos\theta + \underset{\sim}{\hat{e}_2}\sin\theta}$$

By projecting the angular velocities onto axis 1, 2 and 3, which are the body axis. We have:

$$\underset{\sim}{\omega} = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{pmatrix}\dot{\theta}\\ \dot{\phi}\sin\theta\\ \dot{\chi}+\dot{\phi}\sin\theta\end{pmatrix}$$

$$\underset{\sim}{J} = \begin{pmatrix}J_1\\ J_2\\ J_3\end{pmatrix} = \begin{pmatrix}I_1\dot{\theta}\\ I_1\,\dot{\phi}\sin\theta\\ I_3(\dot{\chi}+\dot{\phi}\sin\theta)\end{pmatrix}$$

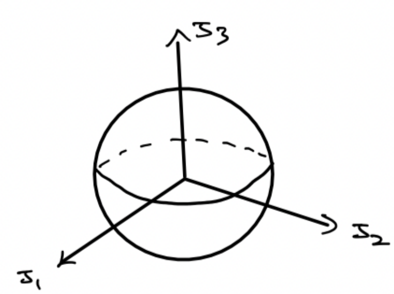
Realising that J_3 ($G_3 = 0$), J_z ($G_z = 0$) and total E is constant,

$$\boxed{\dot{\phi} = \frac{J_z - J_3\cos\theta}{I_1\sin^2\theta}} \quad \text{(space frequency)}$$

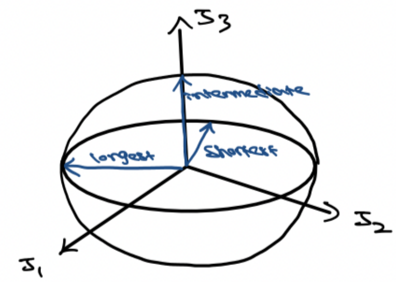
$$\boxed{\dot{\chi} = \frac{J_3}{I_3} - \dot{\phi}\cos\theta} \quad \text{(body frequency)}$$

if we take $\underset{\sim}{J}$ to point along z-axis, we can find the familiar expressions for $\dot{\chi}$ and $\dot{\phi}$.

3.3 Representation in J-space



(a) Conservation of angular momentum



(b) Conservation of energy

Conservation of angular momentum:

$$|\vec{J}|^2 = J_1^2 + J_2^2 + J_3^2 = \text{constant} \quad \text{(sphere)}$$

Conservation of energy:

$$E = \frac{1}{2}\left(\frac{J_1^2}{I_1} + \frac{J_2^2}{I_2} + \frac{J_3^2}{I_3}\right) = \text{constant} \quad \text{(ellipsoid)}$$

The intersection of the sphere and ellipsoid tells you where $\vec{J}$ could point.

3.3.1 If energy is not conserved → Major Axis Theorem

The ellipsoid would shrink until a point where the smallest ellipsoid would fit into the sphere.

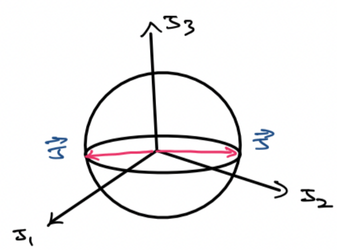


Figure 7: Major axis theorem

The $\vec{J}$ is aligned with the largest moment of inertia, I i.e. the body is spinning about its major axis.

3.3.2 Stability of body

If the object is rotating about minor axis, I_1, at maximum energy state. It is stable. (Minor axis theorem)

If the object is rotating about major axis, I_3, at minimum energy state. It is stable. (Major axis theorem)

If the object is rotating about the intermediate axis, I_2, there are many directions in which $\vec{J}$ can point while conserving energy. Leads to chaotic tumbling. Unstable.

3.4 Applications

3.4.1 Gyroscope - forced precession

Write down energy:

$$E = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}I_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 + mgh\cos\theta$$

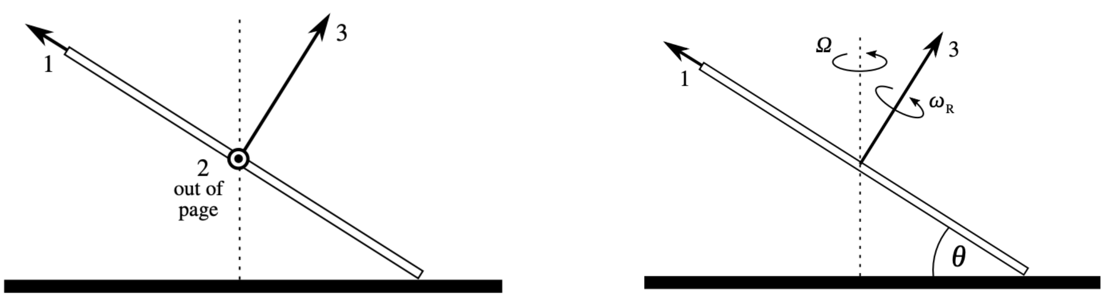
Substitute expressions for $\dot{\chi}$ and $\dot{\phi}$, and find out T and U_{eff}

1. At steady precession, θ is at the equilibrium position (min U_{eff}) and is constant i.e. $\frac{dU_{\text{eff}}}{d\theta} = 0$ and seeking physical solutions for $\dot{\phi}$ gives the gyroscope condition:

$$J_3^2 \gg mghI_1$$

2. Consider gyroscope with horizontal axis supported at one end. Taylor expand U_{eff} about $\theta = \frac{\pi}{2} + \epsilon \rightarrow$ SHM in $\epsilon \rightarrow$ nutation

3.4.2 Spinning disks



1. Rotation about 1-axis only
2. Rotation at Ω about vertical axis + rotation at ω_R about the 3-axis

$$(\omega_1, 0, 0) = (\Omega\sin\theta, 0, \Omega_R + \Omega\cos\theta)$$

So,

$$\omega_R = -\Omega\cos\theta$$

4 Normal modes

Small oscillations of a system about its equilibrium position ($\frac{dU}{dx} = 0$) tend to give a linear equation of motion that is simple harmonic. The solutions are known as **normal modes**, in which **all elements oscillate at a single frequency**. We can decompose a complicated response into the normal modes, evolve them in time and recombine them using the principle of superposition.

Normal modes theorem: N normal modes = M masses $\times$ D dimensions

4.1 Solving for normal modes

Step 1: Set up equation of motion (the physics)

1. Force balance

2. Euler-Lagrange

Put equation of motion into the form:

$$\begin{pmatrix} m\ddot{x}_1 \\ m\ddot{x}_2 \end{pmatrix} = \begin{pmatrix} \text{-} & \text{-} \\ \text{-} & \text{-} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Substitute trial solution, $x_k \propto e^{i\omega t}$:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$$

Step 2: Solve matrix (just math)

$$\underbrace{\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}}_{\text{Det} = 0} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solve for eigenvalues ($m\omega^2$) and eigenvectors (shape of the normal modes).

Normal coordinates

$$m\ddot{q}_1 = -(m\omega_1^2)q_1$$
$$m\ddot{q}_2 = -(m\omega_2^2)q_2$$

Useful graphs to plot

1. $\frac{X_2}{X_1}$ against $\frac{m_2}{m_1}$

2. ω^2 against $\frac{m_2}{m_1}$

4.2 Normal modes generalised

Near equilibrium behaviour:

1. Kinetic energy is a quadratic function of $\underline{\dot{q}}$
2. Potential energy is a quadratic function of q

$$\boxed{E = U_0 + \frac{1}{2}\sum_{ij} \dot{q}_i M_{ij} \dot{q}_j + \frac{1}{2}\sum_{ij} q_i K_{ij} q_j}$$

Setting $\frac{dE}{dt} = 0$ gives a very familiar expression $[m\ddot{x} = -kx]$:

$$\boxed{M_{ij}\ddot{q}_j = -K_{ij}q_j}$$

where $\underline{\underline{\mathrm{M}}}$ is the **mass matrix** (diagonal) and $\underline{\underline{\mathrm{K}}}$ is the **spring constant matrix** (symmetric).

It is an eigenvalue-eigenvector problem at heart:

$$(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) \cdot \underline{q} = 0$$

If there is a forcing term:

$$(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) \cdot \underline{q} = \underset{\sim}{F}$$

Step 1: Write down T and U

Step 2: Find $\underline{\underline{M}}$ and $\underline{\underline{K}}$ by inspection

Step 3: Solve $\boxed{|\underline{\underline{K}} - \omega^2 \underline{\underline{M}}| = 0}$ to find normal modes and characteristic frequencies

Step 4: If there is forcing term, invert the matrix, $\underline{\underline{K}} - \omega^2 \underline{\underline{M}}$, to find $\underset{\sim}{q}$

4.3 General result

Any general free oscillation can be expressed as a linear superposition of simple normal modes. Example:

$$\Theta(t) = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = (A+Bt) \begin{pmatrix} | \\ e_1 \\ | \end{pmatrix} + (C\cos\omega_- t + D\sin\omega_- t) \begin{pmatrix} | \\ e_2 \\ | \end{pmatrix} + (E\cos\omega_+ t + F\sin\omega_+ t) \begin{pmatrix} | \\ e_3 \\ | \end{pmatrix}$$

Given initial & boundary conditions, we can solve for the motion of the body.

4.4 The Physics of normal modes

1. **Stability**
   If all $\omega^2 > 0$, the system is stable.
   If any $\omega^2 < 0$, growing modes exists ($\propto e^{\kappa t}$) and the system is unstable.
   If $\omega^2 = 0$, correspond to rotation and translation of the whole system.

2. **Degeneracy**

5 Elasticity

5.1 Basic definitions

Stress, strain and Young's modulus E:

$$\tau = \frac{F}{A} = Ee$$

Stress τ has the same dimensions as pressure. Strain e is fractional distortion $\frac{\delta l}{l}$.

Poisson's ratio: gives the strain in one direction caused by the stress imposed in an orthogonal direction. For example, stretching in one direction by δl causes compression in the orthogonal direction by $\sigma \delta l$.

Assuming isotropic linear elastic medium, putting everything together gives the **master equation**:

$$E \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & -\sigma & -\sigma \\ -\sigma & 1 & -\sigma \\ -\sigma & -\sigma & 1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$$

in component form: $\boxed{Ee_1 = \tau_1 - \sigma\tau_2 - \sigma\tau_3}$

Bulk modulus B

B is the proportionality constant between pressure P and volume strain.

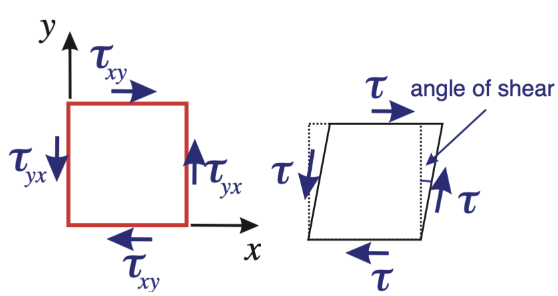
$$\boxed{dP = -B\frac{dV}{V}}$$

Proof: consider medium under isotropic pressure: $\tau_1 = \tau_2 = \tau_3 = -P$

where $\frac{\delta V}{V} \approx (1+e_1)(1+e_2)(1+e_3) - 1 \approx e_1 + e_2 + e_3$ to first-order expansion

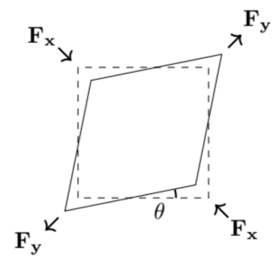
$$\boxed{B = \frac{E}{3(1-2\sigma)}}$$

Shear modulus G



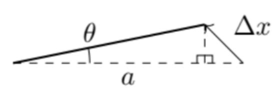
$$G = \frac{\text{shear stress}}{\text{shear angle}} = \boxed{\frac{E}{2(1+\sigma)}}$$

Shear stress and shear strain



Compression and stretching in perpendicular directions lead to shear stress and strain. Equivalent to AC-rotation of θ and shearing by angle 2θ.

Consider a small section:



1. Prove that $\theta = \epsilon_x$

$$\tan\theta \approx \theta = \frac{\Delta x/\sqrt{2}}{a}$$

$$\epsilon_x = \frac{\Delta l}{l} = \frac{2\Delta x}{2\sqrt{2}a} \quad \Rightarrow \quad \theta = \epsilon_x$$

2. Apply master equation (note that $\tau_x = -\tau_y = \tau$):

$$E\epsilon_x = \tau_x - \sigma\tau_y = \tau(1+\sigma)$$

$$G = \frac{\text{shear stress}}{\text{shear angle}} = \frac{\tau}{2\theta} = \frac{E}{2(1+\sigma)}$$

5.2 Formal matrix representation of stress and strain

Stress: force per unit area transmitted across planes in the medium.

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

i.e. τ_{xy} is the force/unit area in the x-direction transmitted through the y-plane (plane perpendicular to y-axis).

The stress tensor is symmetric and can be diagonalised.

Strain:

$$\underline{\underline{e}} = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_z z \end{pmatrix}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i}\right)$$

In general,

1. Diagonal normal components

$$\boxed{E e_{xx} = \tau_{xx} - \sigma\tau_{yy} - \sigma\tau_{zz} \quad \text{etc...}}$$

2. Off-diagonal shear components

$$\boxed{\tau_{xy} = G \underbrace{(2e_{xy})}_{\substack{\text{shear}\\ \text{angle}}} \quad \text{etc...}}$$

3. Matrix representation

$$\underline{\underline{\tau}} = \left(B - \frac{2}{3}G\right) Tr(\underline{\underline{e}})\underline{\underline{I}} + 2G\underline{\underline{e}}$$

5.3 Elastic Strain Energy

$$U = \frac{1}{2}(\tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3)$$

$$U = \frac{1}{2}\left[\left(B - \frac{2}{3}G\right)\ \left(Tr(\underline{\underline{e}})\right)^2 + 2G\,Tr\left(\underline{\underline{e^2}}\right)\right]$$

Simply, it is just:

$$U = \frac{1}{2}C\phi$$

5.4 Problem-solving

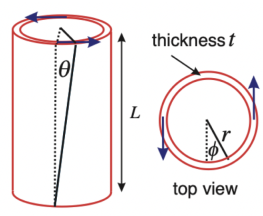
1. Integrating strain $e(\underset{\sim}{r})$

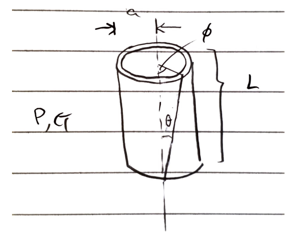
$$\Delta l = \int_l e(\underset{\sim}{r})\ dl$$

2. Force balance between pressure and stress. Use the correct areas for each of them:

$$P \times \text{area}_1 = \tau \times \text{area}_2$$

3. Twisting





$$\text{Relating shear angle and twist angle, } \theta = \frac{r\phi}{L} \quad -(1)$$

Consider a cylindrical shell of radius r and thickness dr,

$$\text{Couple: } dG = F \cdot r \quad -(2)$$

$$F = \text{area} \cdot \text{shear stress} = (2\pi r dr)\, G\theta \quad -(3)$$

$$\therefore \boxed{dG = \frac{2\pi r^3 G\phi}{L} dr}$$

$$G = \int_0^a dG$$

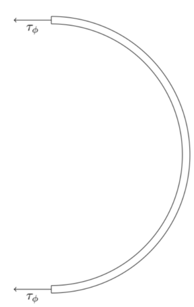
$$W = \int_0^{\theta_0} G \, d\phi$$

4. Bending

$$\text{Torque, } G = B$$

$$W = \int \frac{\phi}{L} EI \, d\phi$$

5. Thin tube - relationship between tangential stress and radial stress.



Consider infinitesimal section of width dr and length l into the page.

$$\text{Force due to tangential stress} = \text{Difference in radial stress}$$

$$2 \times (\tau_\phi \times l\, dr) = d(\tau_r \times 2rl)$$

$$\Rightarrow \tau_\phi = \frac{d}{dr}(r\tau_r)$$

5.5 Beam bending

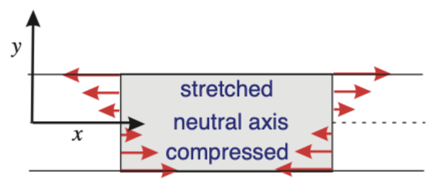
Moment of area (related to beam stiffness):

$$I = \int_A y^2 \, dA = \int_A y^2 \, dxdy$$

Calculate I for different directions.

Derivation of bending moment

Consider a beam subjected to bending moment causing it to bend into an arc with radius of curvature R.



The bending moment is the moment of the longitudinal forces on the element.
First, the strain at distance y from neutral axis is:

$$e_{xx} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

Hence, the total moment across a cross-section of the beam is:

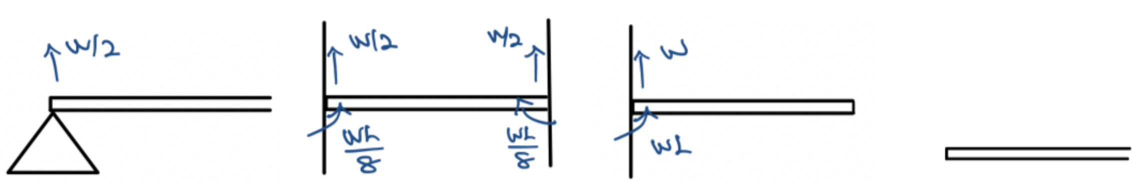
$$B = \int y \times \underbrace{\left(E\frac{y}{R}\right)}_{\text{force/area}} dA = \frac{E}{R}\int y^2 dA$$

$$\boxed{B = \frac{EI}{R} = EIy''}$$

where $\frac{1}{R} = \frac{y''}{(1+y'^2)^{3/2}} \approx y''$ is radius of curvature. (derived by matching gradient and curvature at a point of a curve to a circle of radius R)

Procedure

1. Draw diagram:



(a) Supported (b) Rigidly-clamped (c) Rigidly-clamped (d) Unsupported

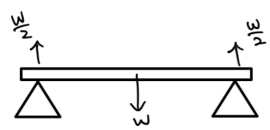
2. Draw force and bending moment graph + define bending moment:

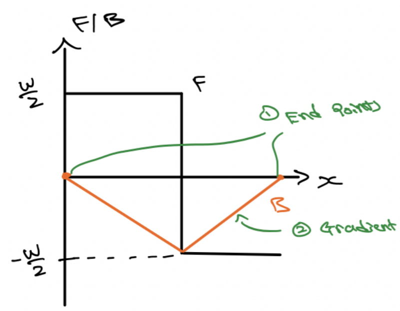
Plotting the force and bending moment graph:
1) Look from left to right, and the force is the prevailing force.
2) Force is the negative of the gradient of B.
3) End points of bending moment.

$$F = -\frac{dB}{dx}$$

Examples:

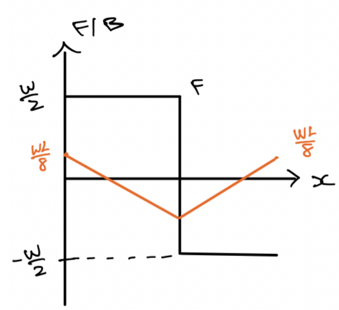
(a) 2 supported ends





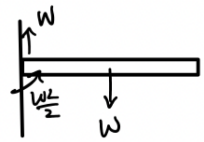
$$B(x) = \begin{cases} -\frac{W}{2}x, & 0 < x < \frac{L}{2} \\ -\frac{W}{2}(L-x), & \frac{1}{2}L < x < L \end{cases}$$

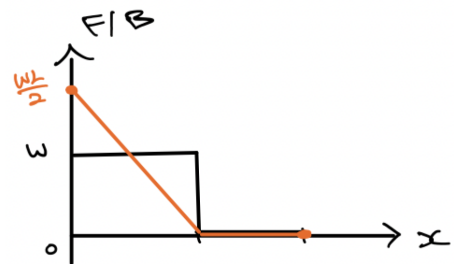
(b) 2 clamped ends



$$B(x) = \begin{cases} -\frac{W}{2}x + \frac{1}{8}WL, & 0 < x < \frac{L}{2} \\ -\frac{W}{2}(L-x) + \frac{1}{8}WL, & \frac{1}{2}L < x < L \end{cases}$$

(c) one clamp, one free, weight in middle





$$B(x) = \begin{cases} -Wx + \frac{1}{2}WL, & 0 < x < \frac{L}{2} \\ 0, & \frac{1}{2}L < x < L \end{cases}$$

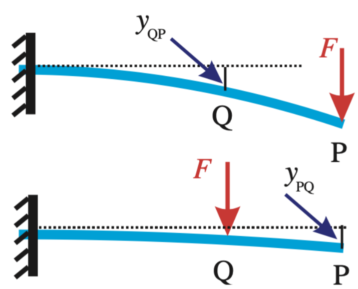
3. Boundary conditions (usually taken at $x = 0$ or $x = L/2$):
   - Unsupported (free): $y'' = y''' = 0$
   - Clamped (cantilever): $y = y' = 0$
   - Supported (hinged): $y = y'' = 0$
   - Symmetry about centre: $y'(L/2) = 0$
4. (Math) Solve for $y(x)$ by integration in:

$$B(x) = EIy''$$

5. Obtain expression for deflection y(x) in terms of x.

5.6 Reciprocity theorem

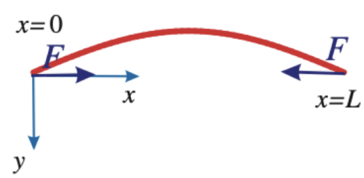
Since the elastic constant is in the linear regime and is independent of time. Order of loading does not matter, the energy has to be the same for the same final configuration.



Since the stored energy when loads are applied at P and Q must be the same whether the load at Q is applied before or after that at P, by an energy argument:

$$y_{PQ} = y_{QP} \quad \text{(can be proven)}$$

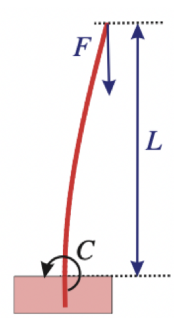
5.7 Buckling - Euler Strut



$$B = -Fy(x) \longrightarrow y'' + \frac{F}{EI}y = 0 \longrightarrow y = A\sin\left(\sqrt{\frac{F}{EI}}\,x\right)$$

Boundary conditions $x = L$, $y = 0$,

$$\boxed{F_E = \frac{\pi^2 EI}{L^2}} \quad \text{(Buckling force)}$$



$$\boxed{F_E = \frac{\pi^2 EI}{4L^2}} \quad \text{(Buckling force)}$$

5.8 Elastic waves

Equation of motion:

$$\rho \frac{\partial^2 X_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j}$$

$$Tr(\underline{\underline{e}}) = \nabla \cdot \underset{\sim}{X}$$

$$\boxed{\rho \frac{\partial^2 \underset{\sim}{X}}{\partial t^2} = \left(B + \frac{1}{3}G\right) \nabla(\nabla \cdot \underset{\sim}{X}) + G\nabla^2 \underset{\sim}{X}}$$

Finding normal modes (wave solutions):
Suppose,

$$\underset{\sim}{X} = \begin{pmatrix} X_0 \\ Y_0 \\ Z_0 \end{pmatrix} e^{i(\omega t - kx)}$$

Substitute into equation of motion.
Boundary conditions:

1. Free boundary: normal component of stress vanish

$$\underset{\sim}{n} \cdot \underline{\underline{\tau}} \cdot d\underset{\sim}{S} = 0$$

2. Fixed boundary: normal component of displacement vanish

$$\underset{\sim}{n} \cdot \underset{\sim}{X} = 0$$

5.9 Application

5.9.1 Normal modes on elastic bar

Assuming the beam is at equilibrium,

Weight per unit length:

$$\boxed{W = EIy''''}$$

Equation of motion:

$$\rho y'' = -EIy''''$$

Boundary condition as detailed above.

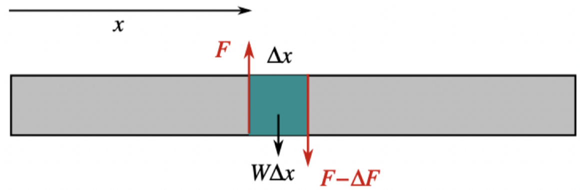
Look for normal modes:

$$y(x,t) = y(x)e^{i\omega t}$$

Solve:

$$EIy'''' - \omega^2 \rho y = 0$$

Derivation for $W = EIy''''$



Consider forces on element of length Δx

1. Balance vertical forces

$$\Delta F = -W\Delta x \quad \Rightarrow W = -\frac{dF}{dx}$$

2. Force F produce a couple which is balanced by change in bending moment

$$\Delta B = -F\Delta x \quad \Rightarrow F = -\frac{dB}{dx}$$

3. Bending moment

$$B = EIy''$$

4. Combining

$$W = EIy''''$$

5.10 Tripos Q & A

Q: Why liquid has isotropic pressure?

A: Liquid cannot sustain shear stress and hence will respond continuously to it. Imagine compressing a square on the left/right and tension on the top/bottom. Look at an inscribed square, those forces actually correspond to shear of the square.

Q: Derive the radial strain e_r and tangential strain e_ϕ in a pipe.

A: Elemental analysis:

$$e_r = \frac{dr + R'dr - dr}{dr} = \frac{\partial R}{\partial r}$$

where R is the radial displacement at radius r.

$$e_\phi = \frac{2\pi(R+r) - 2\pi r}{2\pi r} = \frac{R}{r}$$

6 Fluid Dynamics

6.1 Archimedes Principle

The **upthrust** is equal and opposite to the weight of the fluid it displaced.

$$F_{\text{upthrust}} = \rho g V = \int_V \rho g \; dV$$

6.2 Stresses in liquids and gases

Fluids cannot maintain a shear stress because the molecules move past each other over some timescale t_s. A sudden shear e_{xy} produce a stress τ_{xy} that rapidly decays. Likewise for normal strain and strain.

To maintain stress in fluid, it has to be continuously sheared. (Concept of viscosity)

6.3 Equation of motion → Euler's equation

Consider forces acting on fluid element,

$$F_i = \underbrace{\left(-\Delta x_i \frac{\partial P}{\partial x_i}\right)}_{\Delta P_i} \underbrace{(\Delta x_j \Delta x_k)}_{\Delta A} = -\frac{\partial P}{\partial x_i} \Delta V$$

$$\frac{m}{\Delta V} \times \text{acceleration} = -\nabla P$$

$$\boxed{\rho \frac{D\vec{v}}{Dt} = -\nabla P + \rho \vec{g} \quad \text{(Euler's equation)}}$$

Convective derivative

Velocity is a function of space and time, $\vec{v}(\vec{x}, t)$.

$$d\vec{v} = dt \frac{\partial v}{\partial t} + \sum_i dx_i \frac{\partial v}{\partial x_i}$$

$$d\vec{v} = dt \left(\frac{\partial v}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right)$$

$$\boxed{\frac{D}{Dt} = \underbrace{\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right)}_{\text{non-inertial frame}}}$$

6.4 Conservation laws for fluids (problem-solving)

Ideal fluid = **incompressible** ($\nabla \cdot \mathbf{v} = 0$) and **inviscid** (no viscosity, $\eta = 0$).

Conserve mass, energy flow rate, volume flow rate, momentum flow rate

6.4.1 Conservation of mass → Continuity equation

$$\underbrace{\int_V \frac{\partial \rho}{\partial t}\, dV}_{\substack{\text{rate of change of} \\ \text{mass inside V}}} + \underbrace{\oint_S \rho \vec{v} \cdot d\underset{\sim}{S}}_{\substack{\text{mass flux through} \\ \text{surface S}}} = 0$$

$$\Rightarrow \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right) dV = 0$$

$$\boxed{\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{(continuity equation)}}$$

$$\boxed{\Rightarrow \nabla \cdot \vec{v} = 0 \quad \text{(incompressible flow)}}$$

6.4.2 Conservation of energy flow rate → Bernoulli's equation

$$\text{Energy flow rate} = \underbrace{A_1 v_1}_{\substack{\text{volume} \\ \text{flow rate}}} (\underbrace{\underbrace{\rho_1 \phi_1}_{\substack{\text{gravitational} \\ \text{(gh)}}} + \underbrace{\frac{1}{2}\rho_1 v_1^2}_{\text{kinetic}} + \underbrace{P_1}_{\substack{\text{WD by} \\ \text{pressure}}}}_{\text{Bernoulli's equation}} + \underbrace{u_1}_{\substack{\text{internal} \\ \text{energy}}})$$

For incompressible flow:

$$\boxed{P + \frac{1}{2}\rho v^2 + \rho \phi_g = \text{constant} \quad \text{(Bernoulli's equation)}}$$

If the flow is steady, $\frac{\partial \vec{v}}{\partial t} = 0$, constant along streamline.

If the flow is steady and irrotational, constant everywhere.

6.4.3 Conservation of volume flow rate

$$\boxed{\text{area} \times \text{velocity is conserved}}$$

$$A_1 v_1 = A_2 v_2$$

6.4.4 Conservation of momentum flow rate

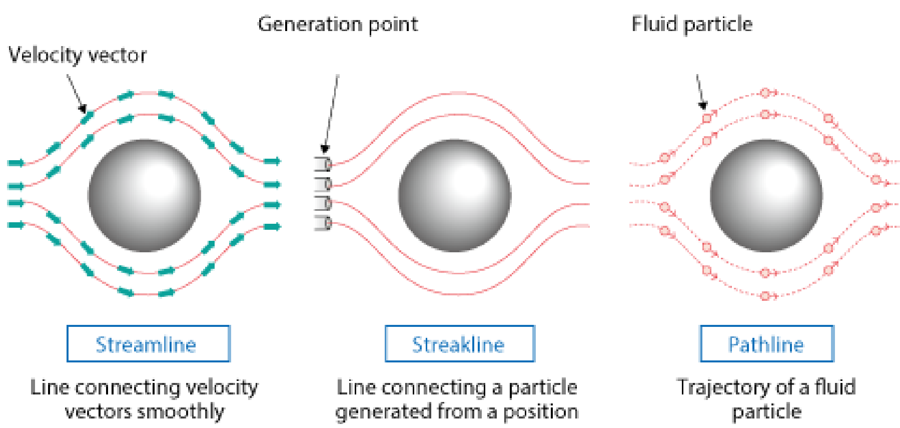
$$\boxed{\text{(mass flow)} \times \text{(momentum/mass) is conserved}}$$

$$(A_1 \rho_1 v_1) \times \vec{v}_1 = (A_2 \rho_2 v_2) \times \vec{v}_2$$

6.5 Visualising fluid flow

Streamline: represents the velocity field. It is tangent to the velocity at any point.

Streakline: connects all points that go through a particular point in space. It is formed by releasing a dye into a fluid at a particular point, giving an instantaneous snapshot of the positions of all the fluid particles that have passed a particular point (least useful but easiest to do experimentally).



6.6 Applications

6.6.1 Flow from water tank

Use Bernoulli's equation to find outflow velocity:

$$P_0 = P_0 - \rho g h + \frac{1}{2}\rho v^2$$

$$\therefore v = \sqrt{2gh}$$

Volume flow rate out is:

$$\boxed{Q = \epsilon v A_{\text{hole}}}$$

where ϵ is the efflux coefficient. For simple hole, $\epsilon = 0.62$. For Borda's mouthpiece, $\epsilon = 0.5$.

6.6.2 Jet of water hitting obstacle

Conserve volume flow rate and/or momentum flow rate to find thickness of water layer.

6.7 Kelvin's circulation theorem

states that the circulation around a loop moving with the fluid is constant. If there is no circulation now, there won't be circulation later.

Consider circulation (amount of rotation) around loop C,

$$K = \oint_C \vec{v} \cdot d\vec{l} = \int \underbrace{\vec{\omega}}_{\nabla \times \vec{v}\text{ (vorticity)}} \cdot d\vec{S} \quad \text{(Stoke's theorem)}$$

$$\boxed{\frac{DK}{Dt} = ... = 0}$$

Implication: vorticity is conserved and moves with the fluid. It cannot be generated in the bulk of the fluid, but only enter the fluid at the boundaries (via boundary layers).

Steady flow & irrotational flow
Steady flow:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}\right) = 0$$

$P + \frac{1}{2}\rho v^2 + \rho\phi_g$ is a constant on a streamline (Bernoulli's principle).

If the flow is steady and irrotational, then $P + \frac{1}{2}\rho v^2 + \rho\phi_g$ is a constant <u>everywhere</u>.

6.8 Potential flow

For irrotational flow, $\nabla \times \vec{v} = 0 \Rightarrow$ conservative, so $\boxed{\vec{v} = \nabla\phi}$ for some scalar field ϕ.
For incompressible flow, $\nabla \cdot \vec{v} = 0$ from continuity equation.

If both are satisfied, the fluid satisfies Laplace's equation:

$$\nabla^2\phi = 0$$

Techniques:

1. Infinite plate: method of images and Green's function.

   Flow past sphere/cylinder: cylindrical and spherical polar coordinates solutions to Laplace's equation

2. Boundary conditions:

$$\text{At boundary } (r = a): \boxed{\mathbf{v}_r = \frac{\partial\phi}{\partial r} = 0}$$

$$\text{As } \ r \to \infty, \ \mathbf{v} = \mathbf{v}_0\,\hat{\boldsymbol{x}} \Rightarrow \boxed{\phi = v_0\,x = v_0 r\cos\theta}$$

3. Find velocity field by:

$$\vec{v} = \nabla\phi$$

4. Apply Bernoulli to find pressure.

5. Force can be calculated from pressure difference ($\Delta P = P - P_0$) by integration. Tip: look at the **squared sinusoidal** term.

$$F = \int \Delta P \, dA$$

A consequence: two sources attract each other. A source and sink repel.
E.g. bubbles merging.

6.8.1 Vortex solutions

Additional vortex term in solution for cylinder

$$\boxed{\vec{v} = \frac{\kappa}{2\pi r} \underset{\sim}{\hat{e}_\theta}}$$

where κ is the strength of rotating vortex and angular velocity is $\omega = \frac{\kappa}{2\pi a^2}$. The velocity of the fluid near the surface is the same as the velocity of the surface of the cylinder.
It has velocity potential:

$$\boxed{\Phi_{\text{vortex}} = \frac{\kappa\theta}{2\pi}}$$

Add together steady flow and vortex:

$$\Phi = V_0 \cos(r + \frac{a^2}{r}) + \frac{\kappa\theta}{2\pi}$$

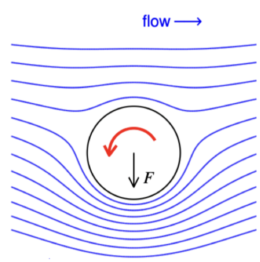


Figure 11: Example: flow past rotating cylinder

Can find v_θ at $r = a$,

$$v_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -2V_0 \sin\theta + \frac{\kappa}{2\pi a}$$

Then can find pressure at the surface using Bernoulli and then force on the cylinder.

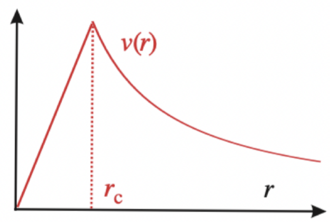
There is a net vertical force per unit length called the **Magnus force**:

$$\boxed{\underset{\sim}{F} = \rho \underset{\sim}{\boldsymbol{V_0}} \times \boldsymbol{\kappa}}$$

Circulation without cylinder - Helmholtz vortex/vortices
Has a core of radius r_c, which rotates as a solid body with vorticity:

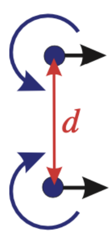
$$\boxed{\Omega = \nabla \times \underset{\sim}{v}}$$

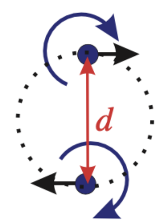
Note that $\omega_{\text{core}} = \Omega/2$.



$$\underset{\sim}{v} = \begin{cases} \frac{\kappa r}{2\pi r_c^2}\, \underset{\sim}{\hat{e}_\theta}, & r < r_c \\ \frac{\kappa}{2\pi r}\, \underset{\sim}{\hat{e}_\theta}, & r > r_c \end{cases}$$

where $\kappa = \pi r_c^2 \Omega$. Analogous to the magnetic field around a thick wire carrying a uniform current.





Two line vortices of opposite sign: blow each other along at speed:

$$v_D = \frac{\kappa}{2\pi d}$$

Two line vortex of same sign: circle each other

6.9 Real fluids

6.9.1 Viscosity

Definition: viscosity is the force per unit area, per unit velocity gradient. Viscosity is the rate of flow of momentum.

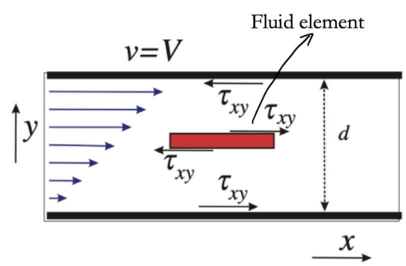
$$\tau_{xy} = \eta \frac{\partial v_x}{\partial y} \qquad \text{(for shear flow in x-direction)}$$

How it arise: when there is spatial variation of velocity.

(a) Sudden shear produce stress $\tau_{xy} = G(2e_{xy})$ that decays over timescale t_s

(b) Continuous shearing give rise to concept of viscosity

Derivation of viscous term:



Viscosity is the proportionality constant between shear stress and rate of shear.

$$\tau_{xy} = \underbrace{\eta}_{\substack{\text{dynamics}\\\text{viscosity}}} \left(2\frac{de_{xy}}{dt} \right)$$

By definition of shear strain,

$$2e_{xy} = \frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x}$$

Differentiate w.r.t. t,

$$2\frac{de_{xy}}{dt} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}$$

So for the case in the diagram with shear flow between $y = 0$ and $y = d$,

$$\tau_{xy} = \eta \frac{\partial v_x}{\partial y}$$

Equation of motion

$$\tau_{ij} = \eta \frac{de_{ij}}{dt} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

Stress variation across volume element:

$$\frac{\partial \tau_{ij}}{\partial x_j} = \eta \left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_j} \right) = \eta \left(\nabla^2 \underset{\sim}{v} + \nabla(\nabla \cdot \underset{\sim}{v}) \right)$$

Additional viscous force term which can be added to Euler equation,

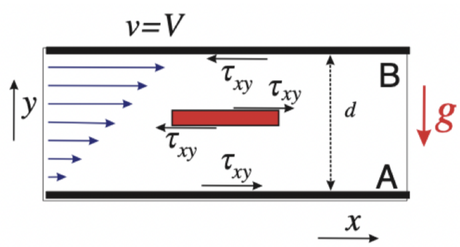
$$\boxed{\underset{\sim}{F}_{\text{viscous}} = \eta \left(\nabla^2 \underset{\sim}{v} + \nabla(\nabla \cdot \underset{\sim}{v}) \right) = \eta \nabla^2 \underset{\sim}{v}}$$

where the second equality is for incompressible fluids.

With the extra viscous term, overall equation of motion* is:

$$\boxed{\rho \frac{D\underset{\sim}{v}}{Dt} = -\nabla P + \rho \underset{\sim}{g} + \eta \nabla^2 \underset{\sim}{v}}$$

Example 1: Viscous shear layer



Steady and uniform (constant cross-section) flow:

$$0 = -\nabla P + \rho \underset{\sim}{g} + \eta \nabla^2 \underset{\sim}{v}$$

Vertical component:

$$\frac{dP}{dy} = -\rho g \tag{1}$$

Horizontal component:

$$\eta \frac{\partial^2 v_x}{\partial y^2} = 0 \Rightarrow v_x = \frac{Vy}{d} \tag{2}$$

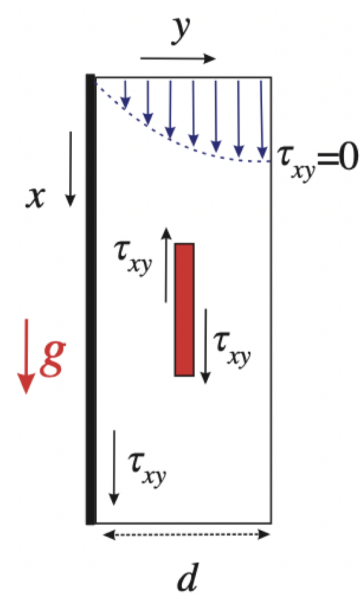
Solve equation (1) and (2), applying no slip boundary condition for equation (2).

Force balance for steady flow: horizontal forces on fluid element are zero, so stresses on either side of element must balance.

$$\tau_{xy} = \eta \frac{dv_x}{dy} = \eta \frac{V}{d}$$

Example 2: Poiseuille flow

1. Draining plate



Force balance: net viscous force on element balance gravity:

$$(Ady)\rho g = d\tau_{xy} \cdot A$$

$$\therefore d\tau_{xy} = -\rho g dy$$

$$\frac{d\tau_{xy}}{dy} = \eta \frac{d^2 v_x}{dy^2} = -\rho g$$

Boundary conditions:

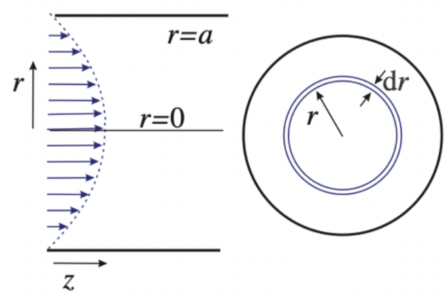
(a) At the wall, $y = 0$, no slip condition (speed of the fluid layer in direct contact with the boundary is identical to the velocity of this boundary) so $v_x = 0$.

(b) At the open end, $y = d$, stress must vanish (no net force on fluid element), so $\tau = 0$.

Obtain parabolic (poiseuille) flow profile.
Can deduce speed at surface and total volume flow rate per unit length:

$$\boxed{Q = \int_0^d v_x dy}$$

2. Pipe



Consider fluid element between r and $r + dr$ with length l
Net pressure force (towards $+z$):

$$F_P = -\frac{dP}{dz}(l)(2\pi r dr)$$

Viscous force on the inside:

$$F_v = \text{area} \times \tau_{zr} = (2\pi r l)\,\eta \frac{dv_z}{dr}$$

Net viscous force (towards $+z$):

$$\frac{dF_v}{dr}\bigg| dr$$

Force balance: **net viscous force = net pressure force**

Boundary conditions: no stress at $r = 0$ and $v_z = 0$ at $r = a$

Integrate to obtain expression for v_z. Can find total volume flow rate:

$$Q = \int_0^a v_z \, 2\pi r dr$$

6.9.2 Reynolds number

$$\boxed{N_R = \frac{\rho v L}{\eta}}$$

where v and L are characteristic velocity and length scales.

Turbulent and laminar flow
Laminar flow: viscous forces dominate. Usually with small dimensions, high viscosity, low velocity and low density.

Boundary condition for surface of solid body = no slip: no radial or tangential velocity.

6.10 Additional information

1. Total kinetic energy of fluid:

$$T = \int_V \frac{1}{2}\rho \mathbf{v}^2 \, dV$$

2. Pendulum immersed in fluid: two effects come into play, the buoyancy force reduces the weight, and the effective mass increases the resistance to acceleration. So,

$$\omega_p' = \sqrt{\frac{g}{l}} \cdot \sqrt{\frac{m_{\text{net gravitational}}}{m_{\text{inertial, effective}}}}$$

$\sim$ End of Notes $\sim$