Physics B - Electromagnetism Summary Notes

By Shikang Ni

Maxwell's equations

$$\text{M1: } \nabla \cdot \mathbf{D} = \rho_{\text{free}} \quad \text{(Gauss' Law)}$$

$$\text{M2: } \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's Law)}$$

$$\text{M3: } \nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss' Law)}$$

$$\text{M4: } \nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{(Ampere's Law)}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad \text{(Lorentz Force Law)}$$

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{(Ohm's Law)}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad \text{(Continuity equation)}$$

where

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0 \mu_r}$$

and

$$\mathbf{P} = \epsilon_0 \chi \mathbf{E} = \epsilon_0 (\epsilon_r - 1)\mathbf{E}$$

$$\mathbf{M} = \chi_m \mathbf{H} = (\mu_r - 1)\mathbf{H}$$

1 Electrostatic field

1.1 Electrostatic force and electric field

$$\vec{F}(\underset{\sim}{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\,\hat{\underset{\sim}{r}}$$

$$\vec{E}(\underset{\sim}{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\underset{\sim}{r}}$$

$$\vec{F} = q\vec{E}$$

1.2 Electric potential and electric field

$$\boxed{\mathbf{E} = -\boldsymbol{\nabla} V}$$

$$V = -\int \mathbf{E}\cdot d\mathbf{l}$$

Electric potential is path independent.
Electric potential at a point r is given by the integral from r to ∞.

Find **E** by Gauss's Law and then find potential by path integration.

1.3 Spatial derivative of E field

Consider line integral of E field around a square loop,

$$\frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$$

$$\Rightarrow \nabla\times\mathbf{E} = 0 \quad \text{(M2 in electrostatics)}$$

By Stokes's Law, line integral of **E** around any closed loop is zero.

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0$$

1.4 Dipole

Electric dipole moment: $\mathbf{p} = q\mathbf{a}$

Dipole potential: $\boxed{V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}}$

Dipole field: $\mathbf{E}(r,\theta) = -\nabla V = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}]$

where

$$\boxed{\nabla = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}}}$$

However, no $\hat{\boldsymbol{\phi}}$ term for no ϕ dependence.

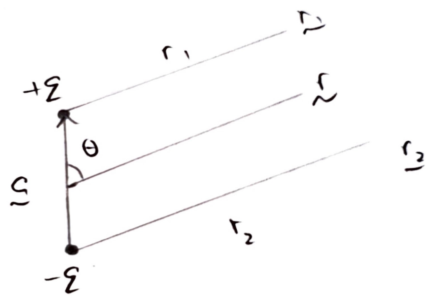
Couple of dipole: $\mathbf{G} = \mathbf{p} \times \mathbf{E} = qE\, a \sin\theta$

Potential energy of dipole in uniform field: $U = -\mathbf{p} \cdot \mathbf{E}$

Force on dipole in non-uniform field: $\mathbf{F} = -\nabla U = \nabla(\mathbf{p} \cdot \mathbf{E})$

1. $\mathbf{p}$ tend to align with $\mathbf{E}$
2. Tend to go towards region of stronger field

Derivation



Superposition principle:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \Rightarrow V = V_1 + V_2$$

At large distances $r \gg a$,

$$r_1 = r - \frac{a}{2}\cos\theta$$

$$r_2 = r + \frac{a}{2}\cos\theta$$

$$V(r_1, r_2) = \frac{q}{4\pi\epsilon_0 r_1} - \frac{q}{4\pi\epsilon_0 r_2}$$

Then proceed with Taylor expansion.

1.5 Continuity equation

Using conservation of charge, consider rate of change of charge enclosed and flux of charge through surface, then divergence theorem:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0}$$

1.6 Gauss's Law

Apply divergence theorem to M1. Use to find electric field $\mathbf{E}$.

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{encl.}}}{\epsilon_0}}$$

Need to know for line charge, sheet of charge and coaxial cable.

1.7 Poisson's equation and Laplace's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \text{(Poisson's equation)}$$

For no charge,

$$\nabla^2 V = 0 \quad \text{(Laplace's equation)}$$

Mathematical methods to solve Poisson's and Laplace's equation.

Uniqueness theorem guarantees that if the solution satisfies the Poisson's equation and the boundary conditions, it is THE solution.

Standard guesses to Laplace's equation

- Cylindrical polar coordinates

$$\phi = Ar\cos\theta + \frac{B\cos\theta}{r}$$

- Spherical polar coordinates

$$\phi = Ar\cos\theta + \frac{B\cos\theta}{r^2}$$

1.8 Conducting sphere in uniform E field

Induced a dipole.

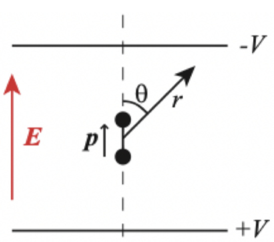


Figure 2.19: Dipole in a uniform field.

1. Write down potential:

$$V = V_{\text{capacitor}} + V_{\text{dipole}}$$

$$V = -E_0\, r\cos\theta + \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

2. Interesting stuff

   - Can find sphere of zero V. Reversing the argument, can find dipole moment of sphere of radius s:

$$p = \underbrace{4\pi\epsilon_0 s^3}_{\text{polarisability } (\alpha)} \quad E_0 = \alpha E_0$$

   - Can find **E** field using $\mathbf{E} = -\nabla V$

1.9 Method of images

The potential V due to charge density $\rho(\mathbf{r})$ in electrostatics obeys the **Poisson's equation**.

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

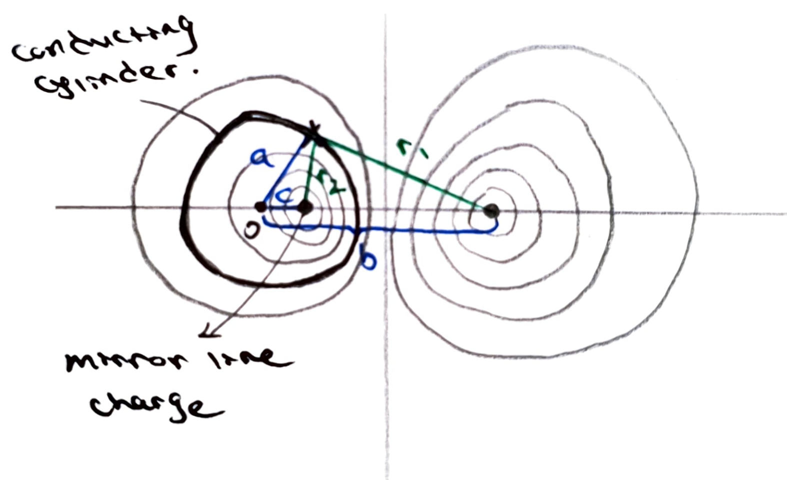
By the **uniqueness theorem**, any solution that satisfies this equation in the domain of interest which also satisfy the Dirichlet or Cauchy boundary conditions is the **only solution**. For Neumann condition, the solution is unique up to a constant.

The method of images works with **high symmetry systems**, where we can **replace the conducting surfaces with mirror charges**, such that the **potential remains constant over the surface** even when the conductor is removed.

Image will have the opposite charge.

Common procedure:
1. Find V
2. Find E using $\mathbf{E} = -\nabla V$

1.9.1 Line charge and conducting cylinder



Modelled as equal and opposite line charges, satisfying (as derived in the Maths course):

$$\boxed{b = \frac{a^2}{c}}$$

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_2}{r_1}\right)$$

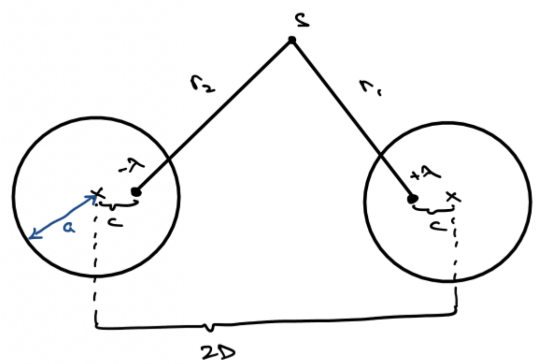
E of individual line charges is found by Gauss's theorem, V can be found soon after. The total V is found by summing them.

It works because the sum of potential due to the line charges is constant on the surface of the cylinder (can be proven). Cylinder is an equipotential surface.

1.10 Capacitance

$$\boxed{C = \frac{Q}{V}}$$

Example: parallel cylinders



Can replace cylinders with line charges at position satisfying $\boxed{a^2 = bc}$ where $b = 2D - c$. Then, find ratio of r_2/r_1 on the line joining them.

1.11 Electrostatic energy

Energy to assemble a system of charges, U_N

$$U_N = \sum_{j=1}^{N} \sum_{i<j} \frac{q_i q_j}{4\pi\epsilon_0 e_{ij}} = \frac{1}{2} \sum_{j=1}^{N} \sum_{i \neq j} \frac{q_i q_j}{4\pi\epsilon_0 e_{ij}}$$

$$\text{Discrete: } U = \frac{1}{2} \sum_{j=1}^{N} q_j V_j$$

where V_j is the potential at position j when all charges are present apart from the j^{th} charge itself.

$$\text{Continuous: } \boxed{U = \frac{1}{2} \int \rho(\mathbf{r}) V(\mathbf{r}) \, d^3\mathbf{r}}$$

1.11.1 Energy stored in capacitor

$$\boxed{U_{\text{capacitor}} = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}}$$

using discrete $U_N = \frac{1}{2}\sum_{j=1}^{N} dQ_j \frac{Q-dQ_j}{C}$

1.11.2 Energy stored in E field

$$\boxed{u_E = \frac{1}{2}\epsilon_0\epsilon|E|^2 = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}}$$

$$U_E = \int u_E \, dV$$

Derivation

1. Consider parallel plate capacitor

$$|E| = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0} \quad \text{and} \quad V = |E|d$$

$$U = \frac{1}{2}QV = \underbrace{\frac{1}{2}\epsilon_0|E|^2}_{u_E} Ad$$

2.
$$U = \frac{1}{2}\int d^3\mathbf{r}\ \rho V = \frac{1}{2}\epsilon_0\int d^3\mathbf{r}\ (\nabla\cdot\mathbf{E})V$$

Using

$$\nabla\cdot(V\mathbf{E}) = (\nabla\cdot\mathbf{E})V + \mathbf{E}\cdot\nabla V$$

$$\frac{1}{2}\epsilon_0\int d^3\mathbf{r}\ (\underbrace{\nabla\cdot(V\mathbf{E})}_{=d\mathbf{S}\cdot V\mathbf{E}} - \mathbf{E}\cdot\nabla V)$$

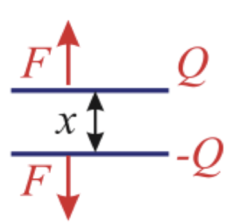
$dS \propto r^2$, $V \propto 1/r$ and $E \propto 1/r^2$. So the term $\to 0$ for large r.

1.12 Virtual work

$$\boxed{F\ dx = \frac{\partial U_s}{\partial x}dx + \frac{\partial U_d}{\partial x}dx}$$

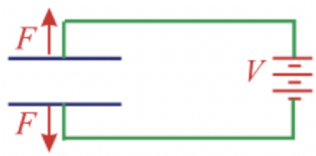
$$dU = dW_{stored} + dW_{mech}$$

where U_s is stored electrostatic energy and U_d is dissipated energy.



1. **Constant Q** (isolated plate) $\rightarrow$ constant **E** (Gauss's law). If x is pulled apart, work is done on system which goes into stored energy $\rightarrow V$ increase.
   Find stored energy $U_s = \frac{1}{2}\epsilon_0|\mathbf{E}|^2 \times Ax$, then equate $\frac{\partial U_s}{\partial x}dx = Fdx$:

$$\boxed{F = \frac{Q^2}{2\epsilon_0 A}}$$



2. **Constant V/p.d.** (connected to external power supply). As x increase, **E** decrease so Q decrease by flowing through the battery. We know $V = |\mathbf{E}|x = \text{const.}$

$$U_s = \frac{1}{2}\epsilon_0|\mathbf{E}|^2 Ax = \frac{1}{2}\epsilon_0\frac{V^2}{x^2}Ax$$

   Find $\left(\frac{\partial U_s}{\partial x}dx\right)$ which is the change in stored energy as the plates are separated. We have to consider also the energy dissipated in power supply as charges flow through it:

$$E = \frac{Q}{A\epsilon_0} \Rightarrow Q = E\epsilon_0 A = \frac{V}{x}\epsilon_0 A$$

   Can find $dQ = \frac{\partial Q}{\partial x}dx$, then we find the energy dissipated:

$$\frac{\partial U_d}{\partial x}dx = -V\,dQ$$

   Using the fact that work done is equal to the change in stored energy and the energy lost, we find an expression for F:

$$\boxed{F = \frac{1}{2}\epsilon_0 V^2\frac{A}{x^2}}$$

   The expression for force whether V is kept constant or Q is kept constant are the SAME!

3. Charge conductor

$$F = |\mathbf{E}|Q = \frac{Q}{2\epsilon_0 A} \times Q$$

1.13 Isotropic dielectric

When an insulator is placed between the plates of a capacitor, held at a constant potential difference, the charge on the plates increases. If the insulator completely fills the space between the plates, the charge increases by a factor ϵ_r:

$$\text{Relative permittivity: } \boxed{\epsilon_r = 1 + \chi}$$

where χ is susceptibility. An applied E field can cause positive and negative bound charge to separate, inducing a dipole moment which is proportional to applied $\mathbf{E}$.

In a uniform field and for homogeneous material, **charge only appears on the external surfaces** due to cancellation of the separated charge within the material.

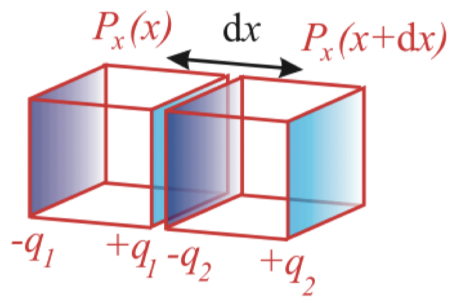


Figure 1: Polarisation in non-uniform field

$$Q = \underbrace{\epsilon_0|\mathbf{E}|A}_{\substack{\text{Field w/o} \\ \text{dielectric}}} + \underbrace{\epsilon_0\chi|\mathbf{E}|A}_{\substack{\text{Additional free charge} \\ \text{to offset} \\ \text{induced bound charge}}} = \epsilon_0(1+\chi)|\mathbf{E}|A$$

$$\text{Dipole moment per unit volume (polarisation): } \boxed{\mathbf{P} = n\mathbf{p} = \epsilon_0\chi\mathbf{E}}$$

$$\text{Bound charge density at surface: } \sigma = |\mathbf{P}_\perp| = \mathbf{P}\cdot\hat{\boldsymbol{n}}$$

$$\text{Polarisation charge density: } \nabla\cdot\mathbf{P} = -\rho_p$$

$$\text{Free charge density: } \nabla\cdot\underbrace{[\epsilon_0\mathbf{E} + \mathbf{P}]}_{\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}} = \rho_{\text{total}} - \rho_p = \rho_f$$

Example, capacitance of parallel plate capacitor.

Boundary conditions for inhomogeneous dielectric

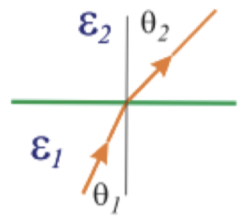
1. Using Gauss's law, and no free charge at boundary.

$$\boxed{D_{1,\perp} = D_{2,\perp}}$$

2. Using the fact that line integral of $\mathbf{E}$ around closed loop is zero.

$$\boxed{E_{1,\parallel} = E_{2,\parallel}}$$

Behaviour of field lines at boundary



By imposing continuity of $E_{\parallel}$ and $D_{\perp}$ at the boundary, we obtain:

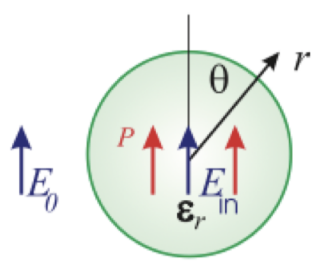
$$\boxed{\epsilon_1 \cot\theta_1 = \epsilon_2 \cot\theta_2}$$

Dielectrics in uniform field

$$\boxed{\mathbf{E}_{\text{in}} = \frac{1}{1+n\chi}\mathbf{E}_0}$$

1. Thin slab perpendicular to field: $n = 1$
2. Cylinder: $n = 1/2$
3. Sphere: $n = 1/3$

Derivation for sphere



Assuming the internal field is uniform. The external field is the original field plus a dipole field generated by surface polarisation charge on the sphere.

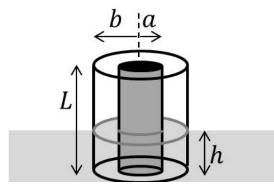
$$V_{in} = -E_{in} r\cos\theta \quad (E_{in} z)$$

$$V_0 = -E_0 r\cos\theta + \frac{\kappa \cos}{r^2} \quad (E_0 z + \text{dipole field})$$

Using $\mathbf{E}(\mathbf{r}) = -\nabla V$ in spherical polar coordinates, $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$ and imposing boundary conditions, we can find find the $\mathbf{E}_{in}$.

1.14 Tripos Q & A

Q1: (2015 Tripos P1 B6) If a constant potential difference V is now applied between the rod and cylinder, find the change in the equilibrium height of the liquid enclosed.



A1: The equilibrium height will be when the total potential energy is minimized. If the liquid rises by Δh, the centre of mass of the excess fluid will rise by $\Delta h/2$. The mass of the excess fluid is $m = \rho \cdot \pi(b^2 - a^2)\Delta h$.

Write down total energy of the system:

$$U = \frac{1}{2}CV^2 + \frac{1}{2}mg\Delta h = \frac{1}{2}CV^2 + \frac{\pi\rho g(b^2 - a^2)(\Delta h)^2}{2}$$

Then, at equilibrium, $F = -\frac{\partial U}{\partial(\Delta h)} = 0$ to find Δh.

Continuity equation

- Charge conservation: the current flux through the surface must be equal to the rate of change of charges within an arbitrary volume.

$$\oint \mathbf{J} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int \rho dV$$

By divergence theorem:

$$\int \nabla \cdot \mathbf{J}\, dV + \int \frac{\partial \rho}{\partial t}\, dV = 0$$

The divergence of a vector field gives the net outflow from the point per unit volume.

So, we obtain the continuity equation:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

-

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

2 Magnetostatic field

Time invariant current and no net charge.

2.1 Magnetostatic force

Force on current element due to magnetic field:

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$$

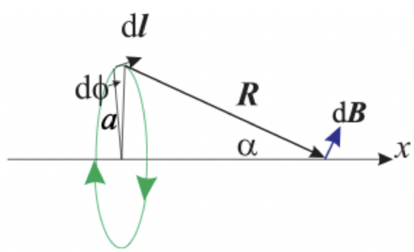
2.2 Biot-Savart Law

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi r^2}\, d\mathbf{l} \times \hat{\boldsymbol{r}}$$

Break down into current elements. Remember **B** is a vector so things might cancel by symmetry. Example: for a current loop, the surviving component is the component along the axis of the loop.

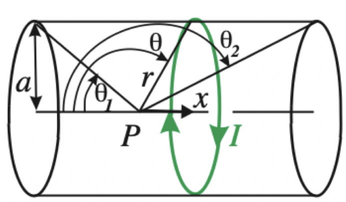
Application: force between two current elements / field on axis of a current loop.

1. On axis of current loop:



$$B_x = a \sin\alpha \frac{\mu_0 I}{2R^2}$$

2. Axis of solenoid:



By integrating B_x and making relevant substitutions, we can obtain the following:

$$B = \frac{\mu_0 I n}{2}(\cos\theta_1 - \cos\theta_2)$$

2.3 Magnetic flux

$$\Phi = \int_S d\mathbf{S} \cdot \mathbf{B}$$

Features of **B** field lines:

1. Form closed loops around current
2. No magnetic monopoles So, zero net flux through closed surface (M3):

$$\nabla \cdot \mathbf{B} = 0$$

2.4 Magnetic dipoles

$$\text{Dipole moment: } \mathbf{m} = \int_S I d\mathbf{S} = I\mathbf{S}$$

$$\text{Couple: } \mathbf{G} = \mathbf{m} \times \mathbf{B}$$

$$\text{Potential energy: } U = -\mathbf{m} \cdot \mathbf{B} = -I\Phi$$

$$\text{Force on dipole: } \mathbf{F} = -\nabla U = \nabla(\mathbf{m} \cdot \mathbf{B})$$

$$\text{Vector potential: } \mathbf{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\boldsymbol{r}}}{r^2}$$

$$\text{Dipole field: } \mathbf{B}_{dip} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{m} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}]$$

In vacuum,

$$\text{Dipole potential: } \boxed{\phi = \frac{\mathbf{m} \cdot \hat{\boldsymbol{r}}}{4\pi r^2}}$$

$$\mathbf{B}_{dip} = -\mu_0 \nabla \phi$$

2.5 Magnetic scalar potential

For current-free regions $\nabla \times \mathbf{B} = 0$,

$$\text{Magnetic field strength: } \mathbf{H} = \frac{\mathbf{B}}{\mu_0 \mu_r} = -\nabla \phi_m$$

$$\boxed{\mathbf{B} = -\mu_0 \nabla \phi_m}$$

For magnetic dipole (similar form as electric dipole):

$$\phi_m = \frac{d\mathbf{m} \cdot \hat{\boldsymbol{r}}}{4\pi r^2} = \frac{I d\mathbf{S} \cdot \hat{\boldsymbol{r}}}{4\pi r^2} = \frac{I d\Omega}{4\pi}$$

where Ω is solid angle with $d\Omega = \frac{d\mathbf{S} \cdot \hat{\boldsymbol{r}}}{r^2}$

For a macroscopic loop:

$$\phi_m = \frac{I\Omega}{4\pi}$$

2.6 Ampere's Law

Apply Stoke's theorem to M4. Use to find magnetic flux **B**.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \mu I$$

$$\oint \mathbf{H} \cdot d\mathbf{l} = I = \int \mathbf{J} \cdot d\mathbf{S}$$

2.7 Magnetic vector potential

Magnetic scalar potential has limited uses and only applies if $\nabla \times \mathbf{B} = 0$ i.e. current-free region. Since M2 tells us that B field is divergence-less, $\nabla \cdot \mathbf{B} = 0$, then $\mathbf{B}$ can be written as a curl of a magnetic vector potential $\mathbf{A}$:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

since Div of Curl is zero i.e. $\nabla \cdot (\nabla \times A) = 0$.

However, $\mathbf{A}$ is undefined to within the addition of a vector $\mathbf{k} = k_x(x)\hat{\boldsymbol{x}} + k_y(y)\hat{\boldsymbol{y}} + k_z(z)\hat{\boldsymbol{z}}$,

$$\mathbf{B} = \nabla \times (\mathbf{A} + \mathbf{k}) = \nabla \times \mathbf{A}$$

But, $\nabla \cdot (\mathbf{A} + \mathbf{k}) \neq \nabla \cdot \mathbf{A}$. The requirement that the curl of $\mathbf{A}$ equals $\mathbf{B}$ does not constrain the divergence of $\mathbf{A}$. We solve this by choosing the gauge which is the process the setting the divergence to some chosen value. Commonly,

$$\nabla \cdot \mathbf{A} = 0$$

In the presence of current sources, we can write a **Poisson's equation** for $\mathbf{A}$.
Starting from M4:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

$$\Rightarrow \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

In components:

$$\nabla^2 A_x = -\mu_0 J_x \quad , \ldots$$

Using Green's function in 3D:

$$\mathbf{A} = \mu_0 \int \frac{\mathbf{J}(\boldsymbol{r'})}{4\pi |\mathbf{r} - \boldsymbol{r'}|} \, d^3\boldsymbol{r'}$$

which calculates the magnetic vector potential at a point by summing over all current sources that contribute to the potential.

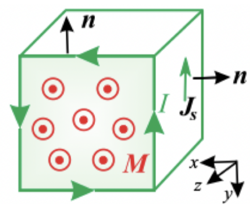
2.8 Magnetic materials

$$\boxed{\text{Magnetisation: } \mathbf{M} = \frac{\mathbf{m}}{V}}$$

$$\mathbf{m}_{\text{total}} = \int \mathbf{M}\, dV$$

$$\text{Magnetisation current density: } \mathbf{J}_m = \nabla \times \mathbf{M}$$

For uniform magnetisation within object, $\mathbf{J}_m$ must reside on the surface:



$$\text{Surface current density: } \mathbf{J}_s = \mathbf{M} \times \mathbf{n}$$

2.9 Magnetic field strength

$$\mathbf{B} = \mu_0(\underbrace{\mathbf{H}}_{\text{due to } \mathbf{J}_{\text{free}}} + \underbrace{\mathbf{M}}_{\text{due to } \mathbf{J}_m})$$

$$\mathbf{M} = \chi_m \mathbf{H} = (\mu_r - 1)\mathbf{H}$$

If $\chi_m < 0$: diamagnetic, $\chi_m > 0$: paramagnetic and $\chi_m \gg 0$, ferromagnetic.

2.10 Boundary conditions

1. Using Gauss's law and M3 that $\mathbf{B}$ is divergence-less (flux):

$$\boxed{\mathbf{B}_{1,\perp} = \mathbf{B}_{2,\perp}}$$

2. Using Stoke's theorem with a closed loop and no free current on the surface:

$$\boxed{\mathbf{H}_{1,\parallel} = \mathbf{H}_{2,\parallel}}$$

If the H/B field approach boundary at an angle, it will "refract". Use these boundary conditions above to derive relevant expressions.

Boundary value problems

$$\boxed{\mathbf{H}_{\text{in}} = \frac{1}{1+n\chi}\mathbf{H}_0}$$

1. Thin slab perpendicular to field: $n = 1$
2. Cylinder: $n = 1/2$
3. Sphere: $n = 1/3$

Problem solving for boundary problems:

1. Noting that $E_\parallel$ and $H_\parallel$ is continuous means that their ratio must be the same:

$$\frac{E_{\parallel,1}}{H_{\parallel,1}} = \frac{E_{\parallel,2}}{H_{\parallel,2}}$$

Forces between currents

- A **current element** (dl) is a infinitesimal piece of wire carrying current I. The vector points in the direction of current flow.

- **Force** on current element due to B field is:

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$$

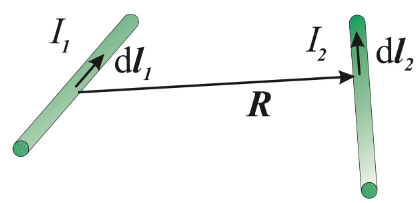
  Consistent with Lorentz force, $d\mathbf{F} = dq\ \mathbf{v} \times \mathbf{B} = \frac{dq}{dt} d\mathbf{l} \times \mathbf{B} = I d\mathbf{l} \times \mathbf{B}$.

- **Magnetic field** produce by current element by Biot Savart Law:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi r^2} d\mathbf{l} \times \hat{\boldsymbol{r}}$$

  The magnetic field lines circulate around the current element.

- Force between two current elements:



$$d\mathbf{B}_2 = \frac{\mu_0 I_1}{4\pi R^2} d\mathbf{l}_1 \times \hat{\mathbf{R}}$$

$$d\mathbf{F}_2 = I_2 d\mathbf{l}_2 \times d\mathbf{B}_2 = \frac{\mu_0 I_1 I_2}{4\pi R^2} d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \hat{\mathbf{R}})$$

- Used to **define the ampere**: the ampere is the current flowing in each of the parallel wire of infinite length and negligible cross section placed 1m apart in vacuum that produces a force of $2 \times 10^{-7} N/m$

3 Electromagnetic Induction

3.1 Faraday's Law (M2)

Faraday's law states that the e.m.f. generated in a circuit is proportional to the rate of change of flux linkage with the circuit. Lenz's law give the proportionality constant as -1.

$$\varepsilon = -\frac{d\Phi}{dt}$$

Relationship between Faraday's Law, Lenz's law and M3
Flux through loop, ϕ is:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

Induced e.m.f. is:

$$\varepsilon = \oint \mathbf{E} \cdot d\mathbf{l}$$

Faraday's law says:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$$

Using Stoke's theorem:

$$\int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = \int_S \dot{\mathbf{B}} \cdot d\mathbf{S}$$

$$\therefore \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(M3)}$$

Derivation

Assume an elemental path is moving at velocity $\mathbf{v}$ in a region of static magnetic field, charges experience a Lorentz force.

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

In the rest frame of the charge, this force appears to be due to an electric field:

$$\mathbf{E} = \mathbf{F}/q = \mathbf{v} \times \mathbf{B}$$

Contribution of length element to the e.m.f. is:

$$d\mathcal{E} = \mathbf{E} \cdot d\mathbf{l}$$

Integrate around the complete loop. Note:

$$\frac{d\mathbf{x}}{dt} \times d\mathbf{l} = \frac{d\mathbf{S}}{dt}$$

3.2 Self-inductance

Definition: self-inductance is the ratio of the flux linked by the system to the current flowing within it.

$$L = \frac{\Phi}{I}$$

where units for Φ is Weber (wb) and for L is Henry (H).

Current changing → induce changing **B** field, whose flux links back to circuit → induce opposite e.m.f. which create opposite flowing current.

*Procedure to find L

1. Find **H**, **B** using Ampere's Law.
2. Find flux Φ using $\int d\mathbf{S} \cdot \mathbf{B}$, surface S.

$$\text{Useful for wires: } \Phi = l \int B \, dr$$

Voltage

$$\mathcal{E} = \oint_{\partial S} \mathbf{E} \cdot d\mathbf{l}$$

$$V_{\text{gap}} = -\oint \mathbf{E} \cdot d\mathbf{l} = \frac{d\Phi}{dt} = \frac{d(LI)}{dt}$$

So,

$$\boxed{V = L\frac{dI}{dt}}$$

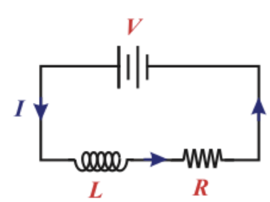
Compare with capacitor:

$$I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

Energy

$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I \quad \text{(energy stored in B field)}$$

Consider L-R circuit,



$$V = IR + L\dot{I}$$

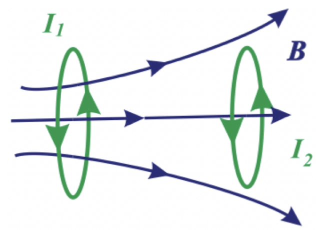
$$VI = I^2R + LI\dot{I}$$

$$P = I^2R + \frac{d}{dt}\left(\frac{1}{2}LI^2\right)$$

Compare with capacitor:

$$U_c = \frac{1}{2}CV^2 \quad \text{(stored energy in E field)}$$

3.2.1 Mutual inductance



$$\boxed{\Phi_2 = \underbrace{M_{21}}_{M_{21}=M_{12}} I_1}$$

$$U_{\text{total}} = \underbrace{\frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2}_{\text{self-energy}} + \underbrace{I_1 I_2 M}_{\text{interaction energy}}$$

Proving that $M_{21} = M_{12}$

Suppose initially $I_1 = I_2 = 0$ and that I_1 is gradually increased, leading to a voltage across the current source:

$$V_{11} = L_1 \frac{dI_1}{dt}$$

Energy stored in magnetic field of the inductor is:

$$U_1 = \frac{1}{2}L_1 I_1^2$$

Now, turn on I_2 while keeping I_1 fixed at its final value. The additional energy stored in the system is:

$$U_2 = \frac{1}{2}L_2 I_2^2$$

At the same time, the increase in I_2 induces a voltage across the first coil:

$$V_{12} = M_{12}\frac{dI_2}{dt}$$

Therefore, energy flows into the magnetic field at a rate:

$$V_{12}I_1 = M_{12}\frac{dI_2}{dt}I_1 = \frac{d}{dt}(M_{12}I_1 I_2)$$

The final energy stored in the system when both currents are at their final value is:

$$U = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12}I_1 I_2$$

The final energy must be the same regardless of the order in which the current sources are turned on, meaning the indices are swapped. Hence,

$$M_{12} = M_{21}$$

3.2.2 Linking L and M

Completing the square for the above expression,

$$U = \frac{1}{2}L_1\left(I_1 + \frac{M}{L_1}I_2\right)^2 + \frac{1}{2}\left(L_2 - \frac{M^2}{L_1}\right)I_2^2 \;\geq\; 0$$

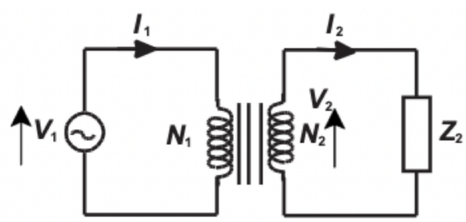
The first expression is guaranteed to be non-negative, hence, the second expression must be non-negative as well. This gives:

$$L_1L_2 \geq M^2$$

$$\Rightarrow \boxed{M = k\sqrt{L_1L_2}}$$

where k is a coupling coefficient between 0 and 1.

3.3 Ideal transformer



A transformer consists of two coils being coupled by means of a core of ferromagnetic material. We assume perfect coupling of flux, wire have zero resistance, linear core and no hysteresis. The primary coil has a sinusoidal voltage, which induces a voltage in the secondary coil.

Since the coils are perfectly coupled, the same flux, Φ, pass through both coils. The linked flux are respectively:

$$\Phi_1 = N_1\Phi \quad \text{and} \quad \Phi_2 = N_2\Phi$$

Faraday's Law:

$$V_1 = \frac{d\Phi_1}{dt} = N_1\frac{d\Phi}{dt}$$
$$V_2 = \frac{d\Phi_2}{dt} = N_2\frac{\Phi}{dt}$$

From this, we obtain the **ratio of voltage** for an ideal transformer:

$$\boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}}$$

Uses: step-down transformer in household appliances to transform socket voltage from 230V to 10s of V needed in appliances. Step-up voltage in power lines to reduce power loss.

Total flux for circuit 1:

$$\Phi_1 = L_1 I_1 - M I_2$$

$$\Phi_2 = -L_2 I_2 + M I_1$$

Differentiate w.r.t. time to obtain V.
Using complex exponential forms of V and I: $V(t) = Ve^{j\omega t}$ and $I(t) = Ie^{j\omega t}$.
Output voltage is constrained by impedance: $V_2 = Z_2 I_2$

Do the math and find an expression for input impedance Z_1 using $M^2 = L_1 L_2$ and $\frac{L_1}{L_1} = \left(\frac{N_1}{N_2}\right)^2$,

$$Z_1 = \frac{j\omega L_1 Z_2 (N_1/N_2)^2}{j\omega L_1 + Z_2 (N_1/N_2)^2}$$

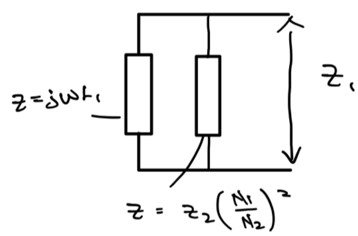


Figure 2: Equivalent circuit for Z_1. Equivalent to $j\omega L_1$ in parallel with $Z_2\left(\frac{N_1}{N_2}\right)^2$

Usually, $\omega L_1 \gg Z_2\left(\frac{N_1}{N_2}\right)^2$

$$\boxed{\textbf{Load impedance ratio}: \frac{Z_1}{Z_2} = \left(\frac{N_1}{N_2}\right)^2}$$

3.4 Energy stored in B field

By considering the total energy of a collection of current loops,

$$U = \frac{1}{2}\int \mathbf{J}\cdot\mathbf{A}\, d^3\mathbf{r}$$

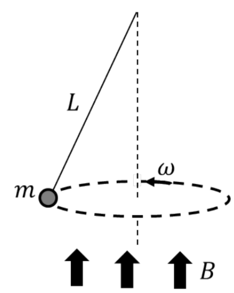
After some mathematical manipulation using vector calculus identity, divergence theorem and M4:

$$\boxed{u_B = \frac{1}{2}\frac{|B|^2}{\mu_0} = \frac{1}{2}\mu_0 |H|^2 = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}}$$

$$U_B = \int u_B\, dV$$

3.5 Tripos Q & A

Q2: A conducting wire of length L supporting a metal sphere of mass m and capacitance C is hung from a grounded rigid pivot and made to rotate as a conical pendulum at angular frequency ω. A uniform magnetic field B oriented vertically is then applied.



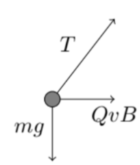
Calculate the charge on the sphere.

A2: Rate of flux cutting:

$$\varepsilon = -\frac{\partial \Phi}{\partial t} = -\dot{A}B = -\frac{1}{2}r^2\omega B$$

$$Q = CV = -C \cdot \frac{1}{2}(L\sin\theta)^2\omega B$$

To find $\sin\theta$, do force balance:



$$T\cos\theta = mg \quad \text{and} \quad F_c = \frac{mv^2}{r} = T\sin\theta + QvB$$

So,

$$\frac{mg}{\cos\theta}\sin\theta + QvB = \frac{mv^2}{L\sin\theta}$$

Substitute in $v = r\omega = \omega L\sin\theta$ and expression for Q. After some approximation and simplification, we can find expression for $\sin^2\theta$, which we can happily substitute back into the initial expression of Q.

4 Electromagnetic waves

4.1 Wave equation

In free space (no free charge and no conduction current), the Maxwell's equations are:

$$\nabla \cdot \mathbf{E} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = -\dot{\boldsymbol{B}} \tag{3}$$

$$\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \dot{\boldsymbol{E}} \tag{4}$$

Derivation of wave equation

1. Take curl of equation (3) and use triple product (curl curl = grad div − div grad)
2. Remove the div term using equation (1)
3. Evaluate curl of **B** using equation (4)

One obtains the wave equation for **E**:

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

*Similar derivations can be done for **B** field.*

Speed of light in free space:

$$\boxed{c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

Solution to the wave equation:

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\nabla \cdot \mathbf{E} = i\,\mathbf{k} \cdot \mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{k} \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = i\,\mathbf{k} \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{k} \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E} = i\omega \mathbf{B} \quad \Rightarrow \quad \boxed{\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}}$$

$$\nabla \times \mathbf{B} = i\mathbf{k} \times \mathbf{B} = -i\omega \mu_0 \epsilon_0 \mathbf{E} \quad \Rightarrow \quad \mathbf{k} \times \mathbf{B} = -\omega \epsilon_0 \mu_0 \mathbf{E}$$

1. TEM nature of electromagnetic wave and :

$$\Rightarrow \mathbf{E} \perp \mathbf{B} \perp \mathbf{k}$$

2. **E**, **B** and **k** is right-handed (**E**uropean **B**roadcasting **K**company)

3. Ratio of E to B gives speed of wave

$$kE = \omega B \Rightarrow \boxed{\frac{E}{B} = v}$$

In dielectric, $\epsilon_0 \to \epsilon_0\epsilon_r$ and $\mu \to \mu_0\mu_r$

Assumptions:
1. Linear $\Rightarrow \mathbf{p} \propto \mathbf{E}$ so $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$
2. Isotropic (same for all directions)
3. Homogeneous (properties uniform throughout)
4. Non-conducting (not metal)

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r\mu_r}} = \frac{c}{n}$$

where $\boxed{n = \sqrt{\epsilon_r\mu_r} = \sqrt{\boldsymbol{\epsilon_r}}}$

Impedance (ratio of E to H)

$$\boxed{Z = \frac{E}{H} = \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}} = Z_0\sqrt{\frac{\mu_r}{\epsilon_r}} = \frac{Z_0}{n}}$$

where Z_0 is the impedance of free space, $Z_0 = 377\Omega$ and μ_r is commonly unity.

Derivation:
Start from $\mathbf{k} \times \mathbf{E}_0 = \omega\mathbf{B}_0$. Then, $k\hat{\boldsymbol{k}} \times \mathbf{E}_0 = \omega\mu_0\mathbf{H}_0$

$$H_0 = \frac{kE_0}{\omega\mu_0} = \frac{E_0}{c\mu_0} = \frac{E_0}{Z_0}$$

4.2 Poynting vector

Poynting vector is a vector which gives the direction, and energy flux (power flow per unit area) of an electromagnetic field:

$$\boxed{\mathbf{N} = \mathbf{E} \times \mathbf{H}}$$

To show that the Poynting vector gives the rate of electromagnetic energy flow per unit area. Consider,

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot \left(-\mu_0\frac{\partial \mathbf{H}}{\partial t}\right) - \mathbf{E} \cdot \left(\epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) - \mathbf{E} \cdot \mathbf{J}$$

Integrating both sides w.r.t. dV and apply divergence theorem,

$$-\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = \int_V \left[\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\mu_0\mathbf{H} \cdot \mathbf{H}\right)}_{\text{magnetic energy density}} + \frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\epsilon_0\mathbf{E} \cdot \mathbf{E}\right)}_{\text{electric energy density}} + \underbrace{\mathbf{E} \cdot \mathbf{J}}_{\text{Energy dissipation by current}}\right] dV$$

The last term is because only **E** field do work.

Other useful expressions:

$$\boxed{N = \frac{|E|^2}{Z}}$$

Time averaged Poynting flux:

$$\boxed{\langle N \rangle = \left| \frac{1}{2} Re[\mathbf{E} \times \mathbf{H}^*] \right|}$$

Maximum rate at which energy is sloshing back and forth at a point:

$$\left| \frac{P}{A} \right|_{\text{max sloshing}} = \left| \frac{1}{2} \text{Im}[\mathbf{E} \times \mathbf{H}^*] \right|$$

4.3 Radiation pressure - force exerted by Poynting vector

The radiation pressure **R** on a surface is the rate of change of momentum per unit area:

$$\mathbf{R} = \frac{\mathbf{N}}{c}$$

If the surface reflects, the radiation pressure is doubled.

Start from energy density:

$$u = \frac{|\mathbf{N}|}{v} = \frac{N\sqrt{\epsilon}}{c}$$

Find energy in given volume:

$$U = u \times V$$

Finding force:

$$F = -\frac{dU}{dx}$$

4.4 Reflections and transmissions

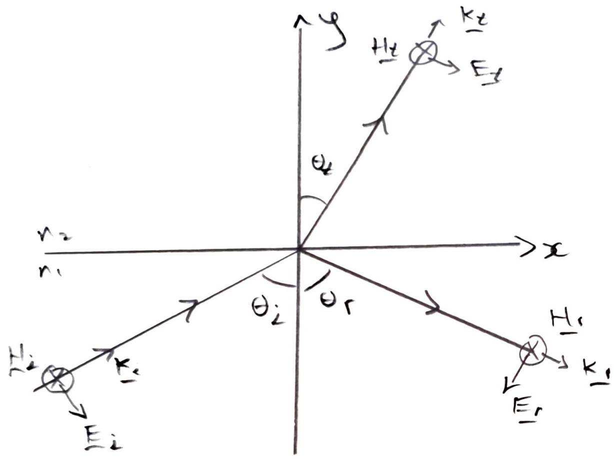


Figure 3: Parallel-polarised plane wave

1. Reflected
   By continuity of $E_{\|}$,
   $$E_{ix} + E_{rx} = E_{tx}$$
   $$E_{i0}\cos\theta_i - E_{r0}\cos\theta_r = E_{t0}\cos\theta_t \quad \text{— (1)}$$
   No phase term,
   $$\Rightarrow \omega_i = \omega_r = \omega_t \quad \text{and} \quad k_i x \sin\theta_i = k_r x \sin\theta_r = k_t x \sin\theta_t$$
   From first and second term, $k_i = k_r$ since same medium, we obtain the reflection law:
   $$\sin\theta_i = \sin\theta_r$$

2. Transmitted
   From first and third term, and using $k = \frac{n\omega}{c}$, we obtain Snell's law:
   $$n_i \sin\theta_i = n_t \sin\theta_t$$
   Furthermore, by continuity of $H_{\|}$ and using $\frac{E}{H} = \frac{Z_0}{n}$,
   $$H_{i0} + H_{r0} = H_{t0}$$
   $$n_1 E_{i0} + n_1 E_{r0} = n_2 E_{t0} \quad \text{— (2)}$$
   From equation (1) and (2) considering $E_{i0} - E_{r0}$ and $E_{i0} + E_{r0}$, we can find expressions for E_{r0} and E_{t0} in terms of E_{i0},
   $$E_{r0} = \frac{\beta - \alpha}{\beta + \alpha} E_{i0}$$

$$E_{t0} = \frac{2}{\beta + \alpha} E_{i0}$$

where

$$\alpha = \frac{\cos\theta_t}{\cos\theta_i}$$

$$\beta = \frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1}$$

Now, playing with the results,

For no reflection, $\alpha = \beta$,

$$\text{Brewster angle: } \tan\theta_B = \frac{n_2}{n_1}$$

For no transmission,

$$\text{Critical angle: } \sin\theta_c = \frac{n_2}{n_1}$$

Fresnel's relations

When talking about polarisation of plane waves, it is always about the orientation of the **E** field,

For parallel-polarised plane wave,

$$r_{\parallel} = \frac{E_r}{E_i} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$t_{\parallel} = \frac{E_t}{E_i} = \frac{2\cos\theta_i}{(n_2/n_1)\cos\theta_i + \cos\theta_t}$$

For perpendicular-polarised plane wave,

$$r_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$t_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i + (n_2/n_1)\cos\theta_t}$$

Power reflection coefficient,

$$\boxed{R = |r|^2 = \left(\frac{n-1}{n+1}\right)^2}$$

Can be derived by considering normal incidence and using small angle approximations.

4.5 Plasma waves

Plasma is a soup of free electrons and positive ions.

Start with:
1. Equation of motion of electron:

$$m_e \frac{d^2\mathbf{r}}{dt^2} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \approx -e\mathbf{E}$$

2. Wave solution of **E**:

$$\mathbf{E} = \mathbf{E}_0 e^{i(kz - \omega t)}$$

Solve for **r**.

3. Relate polarisation to electric field to find expression for χ:

$$\mathbf{P} = n\mathbf{p} = n(-e\mathbf{r}) \quad \text{and} \quad \mathbf{P} = \epsilon_0 \chi \mathbf{E}$$

$$\Rightarrow -ne\mathbf{r} = \epsilon_0 \chi \mathbf{E}$$

where n is the number density of electrons.

4. Derive the relative permittivity in plasma:

$$\boxed{\epsilon = 1 + \chi = 1 - \frac{\omega_p^2}{\omega^2}}$$

where ω_p is the plasma frequency,

$$\boxed{\omega_p^2 = \frac{Ne^2}{\epsilon_0 m_e}}$$

Refractive index for plasma:

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

1. If $\omega < \omega_p$, refractive index is imaginary i.e. $n = i\beta$, find k using $k = n\omega/c$, substitute into expression for **E** to find an exponentially decaying evanescent wave.

For example, reflection of low frequency EM waves off ionosphere enables low-frequency communication.

The magnetic field could also be calculated using $H = \frac{nE}{Z_0}$

2. Can find average and maximum sloshing parts of the Poynting vector.

3. Dispersion relations:

$$v = \frac{\omega}{k} = \frac{c}{n}$$

$$v_g = \frac{d\omega}{dk}$$

4.6 Conducting material

Have to use Ohm's law since there are free currents now,

$$\mathbf{J} = \sigma \mathbf{E}$$

1. From M4:

$$\nabla \times \mathbf{B} = \mu\mu_0 \mathbf{J} + \mu_0\mu\epsilon_0\epsilon \dot{\boldsymbol{E}}$$

and using $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, Obtain,

$$\nabla \times \mathbf{B} = \underbrace{\mu_0\mu\,\sigma\mathbf{E}}_{\text{extra term}} + \underbrace{\mu_0\mu\epsilon_0\epsilon(-i\omega\mathbf{E})}_{\text{dielectric}}$$

2. Rearranging to a similar form as dielectric, we obtain:

$$\mu_0\mu\epsilon_0\left(\epsilon + \frac{i\sigma}{\epsilon_0\omega}\right)(-i\omega\mathbf{E})$$

which looks like an **effective dielectric constant**:

$$\boxed{\epsilon' = \epsilon + \frac{i\sigma}{\epsilon_0\omega} \approx \frac{i\sigma}{\epsilon_0\omega}}$$

3. Find k using $k = \frac{\omega}{c/n}$ where $n = \sqrt{\mu\epsilon'}$ (will have $\sqrt{i}$ this sort of thing)

$$k = (1+i)\sqrt{\frac{\sigma\omega\mu_0\mu}{2}} = \frac{1+i}{\delta}$$

where the **skin depth** is defined:

$$\boxed{\delta = \sqrt{\frac{2}{\sigma\omega\mu_0\mu}} = \frac{1}{\sqrt{\pi\sigma\mu_0 f}}}$$

5. After substituting the expression for k into $\mathbf{E}$,

$$E = E_0 e^{-\frac{z}{\delta}} e^{i(\frac{z}{\delta}-\omega t)}$$

- Amplitude of E field decays by e^{-1} after distance δ (attenuation).
  - The skin depth in metals is usually very small, so EM wave decays very rapidly once they enter a highly conductive material. EM wave can penetrate metals if they are thin enough but it will severely attenuated.

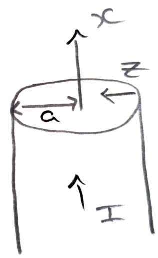
- $\boldsymbol{\delta \propto f^{-1/2}}$, higher frequency waves penetrate shallower. This is why resistance is high for fast oscillating currents.

- $\boldsymbol{\delta \propto \sigma^{-1/2}}$. Greater the conductivity, greater the attenuation, smaller the skin depth.

- $H_y = \sqrt{\frac{\epsilon_0\epsilon'}{\mu_0\mu}} E_x = ... = \sqrt{\frac{\sigma}{2\omega\mu\mu_0}}(1+i)E_x$. Can find poynting vector too.

4.6.1 Skin effect

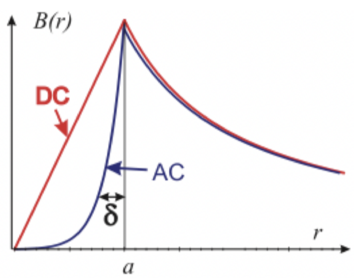
Consider a wire carrying current I that oscillates at frequency ω. Amplitude of current decays away from the surface of the wire.



Using $J = \sigma E$. We obtain an expression for current density as a function of z:

$$J_x = J_0 e^{-\frac{z}{\delta}} e^{i(\frac{z}{\delta} - \omega t)}$$

Amplitude of current density **decays away from the surface** of the wire. Likewise for B field:



Resistance of wire at high frequency:

Qualitative explanation: oscillating currents are confined to the surface of the wire. As the frequency increases, the skin depth gets smaller, and the current is confined to a smaller and smaller region, hence resistance rises.

1. Find current (will be complex) and then an expression for $I^2 = \langle \Re[I]^2 \rangle$:

$$I = \int J_x dS \approx 2\pi a \int J_x(z) dz$$

2. Power dissipated per unit length, P

$$dP = \frac{I^2 R}{L} = (JdA)^2 \cdot \left(\frac{L}{\sigma dA}\right) \cdot \frac{1}{L} = \frac{J^2}{\sigma} dA$$

3. Note that $J^2 = \langle \Re[J_x]^2 \rangle = \frac{J_0^2}{2} e^{-\frac{2z}{\delta}}$. Integrate to find expression for P.
4. Effective resistance per unit length can be found by:

$$R = \frac{P}{I^2} = \boxed{\frac{1}{\sigma} \frac{1}{2\pi a \cdot \delta}}$$

As though the current flows uniformly in a thin shell of thickness δ. $\boldsymbol{R \propto \sigma^{-1/2}}$.

5 Guided waves

5.1 Characteristic impedance

$$Z = \frac{V}{I} = \sqrt{\frac{L}{C}} \quad \text{and} \quad v = \frac{1}{\sqrt{LC}}$$

In an dielectric,

$$Z' = \frac{Z}{n} \quad \text{and} \quad v = \frac{c}{n}$$

Know how to derive C and L for parallel wire, coaxial cable and strip transmission line.

For strip transmission line:

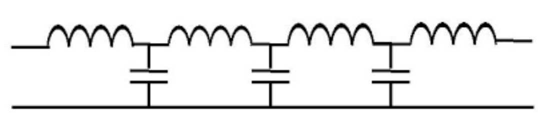
$$C = \frac{a\epsilon\epsilon_0}{d} \quad \text{and} \quad L = \frac{\mu_0 d}{a}$$

Finding impedance
Consider harmonic wave of the form:

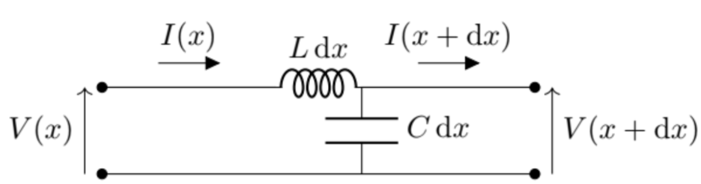
$$I = I_0 e^{i(\omega t - kx)} \quad \text{and} \quad V = V_0 e^{i(\omega t - kx)}$$

$$Z = \frac{V}{I} \quad \text{and} \quad v = \frac{\omega}{k}$$

5.2 Circuit of transmission line



A transmission line can be thought of as a series of inductance with capacitance parallel across the lines.



Using $V_L = L\dot{I}$ and $Q = CV_C \Rightarrow I = C\dot{V}_C$:

$$\text{voltage across inductor:} \quad \frac{\partial V}{\partial x}dx = -Ldx\frac{\partial I}{\partial t} \Rightarrow \boxed{\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}}$$

$$\text{current across capacitor:} \quad \frac{\partial I}{\partial x}dx = -Cdx\frac{\partial V}{\partial t} \Rightarrow \boxed{\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}}$$

Differentiating and combining,

$$\frac{\partial^2 V}{\partial x^2} = -L\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial x}\right) = LC\frac{\partial^2 V}{\partial t^2}$$

Similarly,

$$\boxed{\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}}$$

So, voltage and current wave travels along transmission line at speeds of:

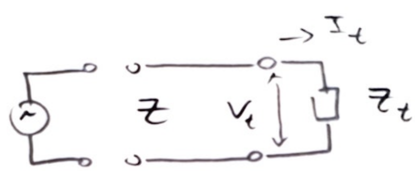
$$\boxed{\text{wave speed: } v = \frac{1}{\sqrt{LC}}}$$

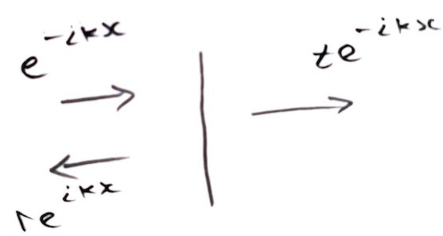
Assuming the solution is of the form $V = V_0 e^{-i(kx-\omega t)}$ and $I = I_0 e^{-i(kx-\omega t)}$:

$$kV = \omega L I \Rightarrow \frac{V}{I} = \frac{\omega L}{k} = Lv$$

$$\boxed{\text{characteristic impedance: } Z = \frac{V}{I} = \sqrt{\frac{L}{C}}}$$

5.3 Power flow on transmission line





Impose continuity conditions:

1. Continuity of voltage: $V_i + V_r = V_t$

$$1 + r = t$$

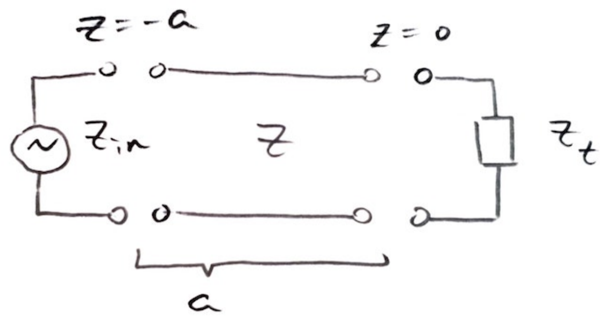
2. Continuity of current: $I_i + I_r = I_t$

$$\frac{1}{Z} - \frac{r}{Z} = \frac{t}{Z_t}$$

Reflection and transmission coefficients can be by solving for r and t in the simultaneous equation:

$$\boxed{r = \frac{Z_t - Z}{Z_t + Z}}$$

$$\boxed{t = \frac{2Z_t}{Z_t + Z}}$$



We have $V_i = V_0 e^{-ikz}$ & $I_i = \frac{V_i}{Z}$ and $V_r = rV_0 e^{+ikz}$ & $I_r = -\frac{V_r}{Z}$ We find an expression for input impedance,

$$Z_{in} = \left.\frac{V_i + V_r}{I_i + I_r}\right|_{z=-a} = \frac{e^{ika} + re^{-ika}}{e^{ika} - re^{-ika}} Z$$

Substituting in the expression for r:

$$\boxed{\frac{Z_{in}}{Z} = \frac{Z_t \cos ka + iZ \sin ka}{Z \cos ka + iZ_t \sin ka}}$$

Some example results,
For open circuit, $Z_t \to \infty$:

$$\frac{Z_{in}}{Z} = -i \cot ka$$

For quarter-wavelength line, $a = \lambda/4$

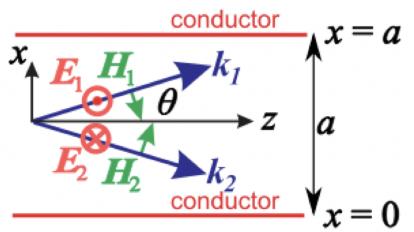
$$Z^2 = Z_{in} Z_t$$

Impedance matching

If a transmission line is terminated with a load equal to the characteristic impedance i.e. impedance is matched, no power will be reflected.

load impedance = transmission line characteristic impedance

5.4 Wave guides



Plane wave pairs:

$$\mathbf{k}_1 = (k \sin\theta, 0, k\cos\theta)$$
$$\mathbf{k}_2 = (-k \sin\theta, 0, k\cos\theta)$$

Then,

$$\begin{aligned} E_y &= E_0[e^{i(\mathbf{k}_1\cdot\mathbf{r})} - e^{i(\mathbf{k}_2\cdot\mathbf{r})}]e^{-i\omega t} \\ &= E_0 e^{i(kz\cos\theta-\omega t)}[e^{ikx\sin\theta} - e^{-ikx\sin\theta}] \\ &= e^{i(kz\cos\theta-\omega t)} \times 2i\sin(kx\sin\theta) \end{aligned}$$

Boundary conditions: electric field has no tangential component at $x = 0$ and $x = a$.

There is **propagating wave in z-direction** and **standing wave in x-direction**.

<u>In z-direction</u>:

$$\text{Phase velocity: } \boxed{v_{ph} = \frac{\omega}{k_g}} \quad \text{where} \quad k_g = k\cos\theta$$

$$\Rightarrow V_{ph} = \frac{c}{\cos\theta}$$

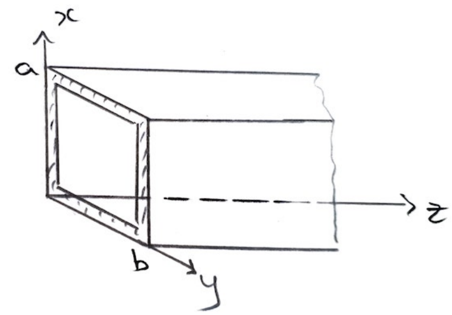
Relationship between phase and group velocity:

$$\boxed{v_{ph}v_{grp} = c^2}$$

<u>In x-direction</u>, using **boundary condition** that electric field is zero at plate,

$$k\sin\theta = \boxed{k_x = \frac{m\pi}{a}}$$

$$k_g^2 = k^2 - k_x^2 = k^2 - \frac{m^2\pi^2}{a^2}$$



Key results:

$$\boxed{k_0^2 = \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_g^2}$$

$$k_z^2 = k_g^2 = \frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}$$

where $m, n \geq 0$ but cannot have $m = n = 0$ at the same time.

$$\text{waveguide equation: } \boxed{k_g^2 = k_0^2 - k_c^2}$$

Also, multiplying by c^2 gives:

$$\boxed{\omega_g^2 = \omega_0^2 - \omega_c^2}$$

For propagating modes, k_g^2 must be positive which gives a cut-off frequency by equating $k_0 = k_c$. Convert to units of Hz for cutoff frequency.

$$\frac{\omega^2}{c^2} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$$

$$\boxed{f_c = \frac{ck_c}{2\pi} = c\left(\frac{m^2}{4a^2} + \frac{n^2}{4b^2}\right)^{1/2}}$$

5.4.1 Rectangular wave guides

TE: Transverse Electric. Lowest possible mode is TE_{n0} meaning there is one-half wavelengths of E-field in the x direction and no variation in the y-direction. The electric field in the direction with no charges can terminate on charges in the metal walls (using $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon\epsilon_0}$).

TM: Transverse Magnetic. Lowest possible mode is TM_{11}. If m=0, n=1 or m=1, n=1. Magnetic field cannot terminate just terminate at the wall since there is no magnetic monopole ($\nabla \cdot \mathbf{B} = 0$). Would not satisfy boundary conditions if that's the case.

~ End of Notes ~